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A COURSE  
OF  
GEOMETRICAL DRAWING,  
CONTAINING  
PRACTICAL GEOMETRY,  
INCLUDING THE USE OF  
DRAWING INSTRUMENTS,  
THE CONSTRUCTION AND USE OF SCALES, ORTHOGRAPHIC PROJECTION,  
AND  
ELEMENTARY DESCRIPTIVE GEOMETRY.

BY  
W. S. BINNS, M.C.P.,  
AUTHOR OF "AN ELEMENTARY TREATISE ON ORTHOGRAPHIC PROJECTION AND  
ISOMETRICAL DRAWING." "A TREATISE ON THE ANALYSIS OF SENTENCES."

PART I.

NEW EDITION, CONSIDERABLY ENLARGED.



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TO  
JOSEPH PRESTWICH, ESQUIRE, M.R.C.S.,  
FOR WHOSE KIND AND UNREMITTING ATTENTION  
DURING THE  
FATAL ILLNESS OF THE AUTHOR'S ONLY SON,  
WHICH TOOK PLACE  
WHILE THIS EDITION OF HIS WORKS WAS PREPARING  
FOR THE PRESS,  
THESE VOLUMES ARE DEDICATED  
WITH LIVELY FEELINGS OF GRATITUDE AND RESPECT  
BY HIS SINCERE FRIEND,  
W. S. BINNS.



## PREFACE.

---

IN the Report for January, 1859, on the Military Examinations, Captain Binney, referring to the deficiency in Geometrical Drawing, exhibited by the majority of the candidates, assigns as one reason, "the absence of *any English work* treating of the subject of Geometrical Drawing generally in anything like a practical manner." An attempt has been made in the present volume to supply, to some extent, the desideratum here pointed out.

While the subject of Plane Geometry enters into the course of study adopted by most of our educational establishments, Descriptive Geometry has hitherto not met with that attention, which the importance of the subject demands. Apart from its practical application to the arts and sciences, Descriptive Geometry ought to recommend itself to our consideration were it only as a means of mental culture. It will be conceded by those who are conversant with the subject, that, in this respect, its value is little, if any, inferior to that derived from the study of Plane Geometry.

Such being the case, and considering its varied and extensive application, that it should not have been more generally recognized as a branch of elementary education, is remarkable. The explanation of such a state of things may, perhaps, be found in the barrenness of English

scientific literature in works on this interesting and useful branch of study. While on the Continent much has been written on the subject, few publications have appeared in England (and those chiefly translations) calculated to serve as text books for the general student. The consequence is that, while in the schools of other countries it is almost universally taught, in our own it is comparatively little known.

In the following pages, the object of the Writer has been to furnish as much practical information on the subject of Geometrical Drawing generally, as the limits of the work would allow; and he is not aware of the existence of any one English work in which the same amount of a similar character of information is to be found as is here given.

It is intended to publish a sequel, embracing Orthographic and Isometric Projection, Perspective, and the Projection of Shadows, as applied to the delineation of solids, the whole forming a general application of the principles inculcated in the present volume.

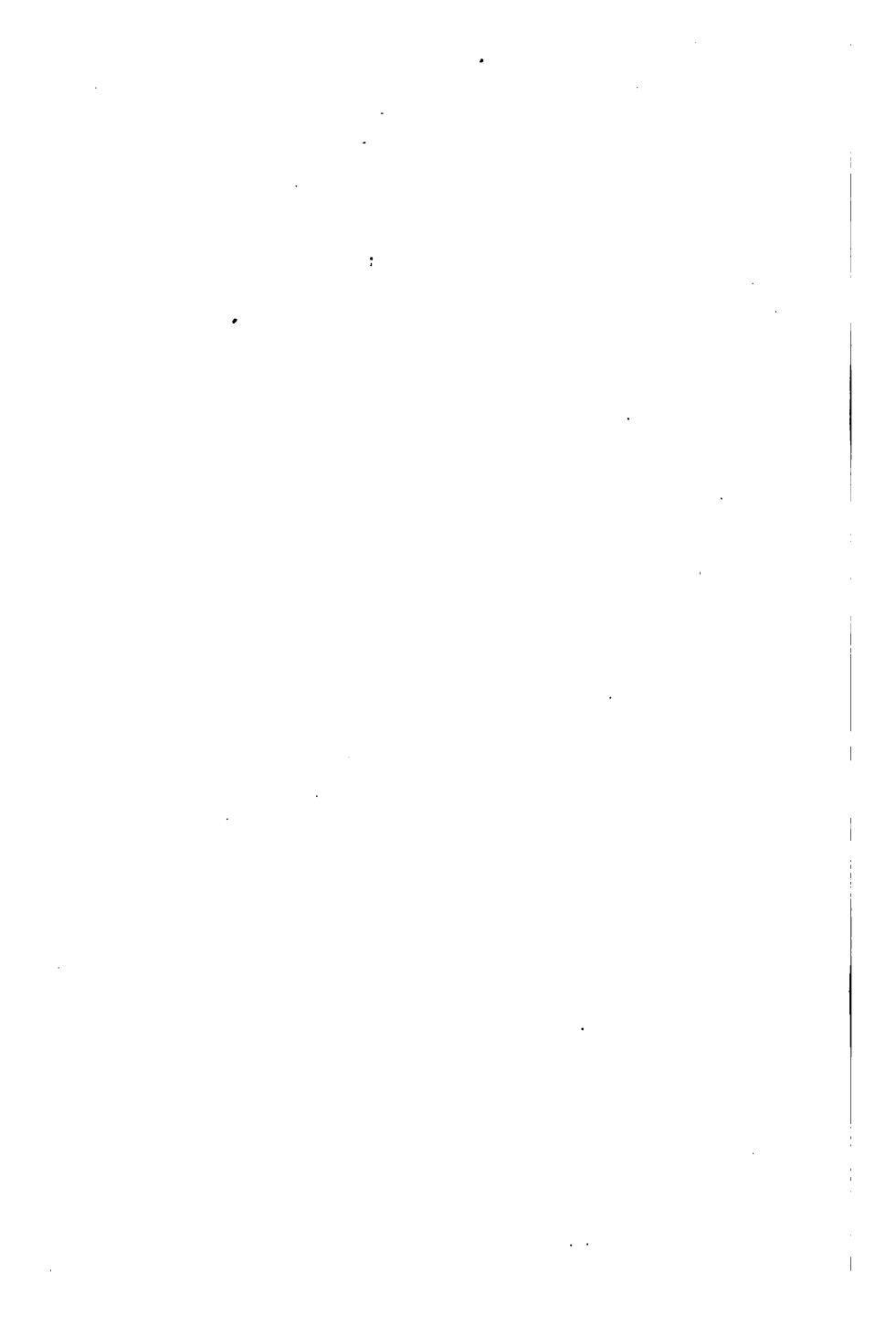
The following are some of the works which have been consulted:—The Woolwich Papers on Geometrical Drawing; Monge's *Géométrie Descriptive*, etc.; Sonnet's *Géométrie Théorique et Pratique*, etc.; Johnson's Translation of the Book of Industrial Design, by M. Armengaud, Ainé, and MM. Armengaud, Jeune, and Amouroux, etc.

## PREFACE TO SECOND EDITION.

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THIS Edition is enlarged by forty pages and forty-one diagrams. The new matter has special reference to the Science Examinations.

It is hoped that the very considerable addition thus made, without increasing the cost of the book, will not only render it one of the cheapest publications on the subject, but will enlarge the usefulness of the work, and secure the continuance of the approval of those numerous Teachers, who have already kindly appreciated the effort thus made to assist them in this important department of their work.



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## CRITICAL NOTICES.

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"We have already had occasion to notice the First Part of this work with commendation, on its appearance about a year ago. Mr. Binns, in the present volume, takes up the subjects of Projection, Perspective, Shadows and Shadow Lines, Shading and Colouring—the whole forming a very practical and complete treatise on Geometrical Drawing. Some of the problems are necessarily of a technical character; but the student who possesses an elementary knowledge of Geometry will find no difficulty in following Mr. Binns's solutions and explanations. We have no doubt that the work will take its place as one of our best text-books on Geometrical Drawing. We are bound to say that it is the best we have seen on the subject."  
—*Educational Times*.

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"Mr. Binns has published, in a single volume, the two little treatises issued by him as aids to the student in Geometrical Drawing. The result is highly satisfactory, for the author has succeeded thoroughly in producing a text-book, which, for practical utility and simplicity, is not excelled by any work of the kind, either in the English or French language, addressed to the classes for whom the present volume is intended.

"The First Part is written and arranged in a style suited to beginners, the Second Part to the more advanced students. The collection of problems is large, and the selection excellent. Mr. Binns's book deserves to be extensively known and patronised."—*The Artizan*.

---

"We have seen several treatises on Geometrical Drawing in French and German superior to the book now before us; but, taken on the whole, it is one of the best works—if not *the* best work—that has yet appeared in the English language. It is true there is no comparison between this book, as regards depth and completeness of treatment of the subject, and another work in German now before us, a work used in the Polytechnic School of Vienna. Any book however must be adapted to the public that is to receive it; a book like this one of Mr. Binns' has been long wanted in England, and appears—apart from some slight deficiencies—to well fulfil its object."—*The Civil Engineer and Architect's Journal*.

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"This is a very valuable book, not only to the student of Geometrical Drawing, but to every student of mathematical science. The

portion devoted to Practical and Descriptive Geometry is full and accurate, and the description of mathematical instruments and their uses of great value. The chapter on Orthographic Projections is very instructive, and the problems given judiciously selected. Instead of falling into the error so frequently committed by writers, of supposing that the student has made a considerable acquaintance with the subject handled, Mr. Binns very wisely assumes that he is addressing those who require to be led on step by step as carefully as Euclid elaborates the geometrical structure, which has been for ages associated with his name. The rules given are simple, clear, and easy of application, so that every student possessing the requisite quality of perseverance may follow the path pointed out by Mr. Binns with pleasure and profit. Those portions of the book which treat of Descriptive Geometry as applicable to engineering are specially valuable, and we can recommend the book with every confidence to all who are anxious to gain a full knowledge of an art so useful as Geometrical Drawing."—*Oldham Chronicle*.

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"To many of our readers, a knowledge of the principles of Geometrical Drawing may be desirable and useful; and through the desideratum now supplied by Mr. Binns, this knowledge may readily be obtained, even by the unaided efforts of the intelligent student. There is ample evidence that the work is not only the best, but the only complete and really practical one of its class in the English language."—*The Electrician*.

---

"Mechanical Drawing, as a branch of education, has hitherto been, to a very great extent, neglected or overlooked in the schools and colleges of Great Britain and Ireland. The reason for this may, perhaps, be found in the fact that we are pre-eminently a practical people: the idea conceived by the engineer or the mechanic is at once commenced with on the large scale, and receives practical development under the guiding hand and eye of the designer, without having previously been committed to paper. This, it need scarcely be said, is not now so much in vogue as it was a few years ago; but notwithstanding the improvement we have undoubtedly made in this respect, we lack yet, in a surprising degree, that familiarity with mechanical drawing which should most certainly be associated with our well-known character as a mining and manufacturing nation.

"Mr. Binns' book is a Course of Geometrical Drawing, and includes Practical Geometry, the use of Drawing Instruments, the Construction and use of Scales, Orthographic Projection, and Elementary Descriptive Geometry. The work is in two parts, the

former comprising the heads we have just referred to, and the latter contains Orthographic, Isometric, and Perspective Projection, the Projection of Shadows, the Principles of Shading, and the Practice of Making Finished Drawings. Mr. Binns has given us a good ground of work of Geometrical Drawing, and we can recommend the book to our readers as being one of very considerable practical utility.....The book in question is the best, and in point of plan and text the most simple that we have any acquaintance with ; and we congratulate Mr. Binns on having been so successful in the authorship of a work which was certainly very much required.”—*The Mining Journal*.

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“Mr. Binns has evinced sufficient care in providing material for this work.”—*Army and Navy Gazette*.

---

“We have before us the revised edition of this little volume, in which the author’s careful observation and researches have been set forth with a great amount of diligent attention. To the increasing class of young men who have to study the practical application of geometry to the delineation of the various machines and structures now being fabricated in all parts of the civilised world, but especially in our own country, this little work will prove of much value. It is divided into two parts, the former of which may be read by any one possessing a knowledge of only the ordinary rules of arithmetic. Commencing with ordinary definitions, it proceeds to sound rules in regard to the treating of drawing material, construction of Scales, after which the student is introduced to several interesting problems in Practical Geometry, merging from this into Orthographic Projection and Descriptive Geometry. The Second Part is an admirable adjunct to the former, and we should strongly recommend those who have studied the First to lose no time in reading the Second, containing as it does all the most important problems in Orthographic and Isometric Projection, Perspective, Shadows, and Colouring. We are of opinion that Mr. Binns has been very successful in the manner he has compiled his work, and feel sure it will be much prized by those students who are preparing for examinations on the subjects it includes. The volume is fairly illustrated by woodcuts, and a welcome addition to the branch of science it treats.”—*The Practical Mechanic’s Journal*.

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“Mr. Binns has done good service to the cause of education in drawing by the publication of these volumes, which, for the first time, afford the student a means of thoroughly grounding himself in the principles of the geometrical representation of objects.

"Mr. Binns has explained minutely the projection of a great number of simple objects, orthographically, on parallel and inclined planes, the construction of sections, the intersection and development of surfaces, and the principles of isometric and perspective representation. More cursorily he has treated the art of shading and colouring."—*The Educator*.

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"Now that our Government science classes are attracting the attention of the artisans of this country, and are increasing and ramifying through our towns and cities, we can fancy that this manual will become very popular with these Government teachers, having the double recommendation of being both cheap and excellent."—*Rochdale Observer*.

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"We have devoted considerable space to this volume, because it deals very thoroughly and well with a topic of great interest in relation to the maintenance of our technical position, and further because such large and successful classes are studying the subjects treated in Bolton, Manchester, and numerous places in the country, and, we might add, throughout the three kingdoms: none can doubt the practical value of the study or the advantage to the student offered by such a volume as Binns's Practical Geometry and Perspective."—*Bolton Guardian*.

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"The book is the best we have seen on the subject of Geometrical Drawing."—*Bury Guardian*.

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"Mr. Binns's Course of Geometrical Drawing is a book so thoroughly appreciated by all who know it, that it no longer needs a critical notice to recommend it, the only thing now required to bring the work into wider circulation being that of placing it into the hands of every practical student. Compared with the many books which have appeared within the last few years upon the subjects treated in this volume, we do not hesitate to say, from practical acquaintance, that, for real utility in the hands of a student, and for clearness of definition and simplicity in direction, it stands unequalled. . . . . The division on Perspective occupying over 40 pages, is in itself worth more than the price of the book. In this subject, admitted to be a difficult one to teach to students, the language is so simple, and the steps so progressive, that no one can fail, with an ordinary share of intelligence and attention, to master the subject."—*Educational Reporter and Science Teachers' Review*.

*From CAPTAIN BINNEY, R.E., Late Military Examiner in Geometrical Drawing.*

"I am very sorry if I have occasioned you any inconvenience by not having sooner sent you any remarks on the Second Part of your work on Descriptive Geometry, etc.

"I think this volume has fully borne out the promise of the First, and while feeling that there are, in both Parts, some details of arrangement and construction, which are capable of improvement; I have yet much pleasure in being able to say, that as far as my judgment goes, it is, as a whole, the most complete and practical work yet written in this country on the very important subject of Geometrical Drawing."

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*From CAPTAIN LENDY, Director of the Practical Military College, Sunbury, Author of a "Treatise on Fortification," "Treatise on Surveying," etc.*

"After a perusal of the Second Part of your Course of Geometrical Drawing, I can only say that I firmly believe it will admirably answer the purpose for which it is intended. The subject has been ably treated in France and Germany, but I do not know any better book published in this country. I might almost say that it is the only one. Its size is convenient, the examples are well selected, and the price is moderate. It will undoubtedly become popular among the students reading for the various military examinations."

---

*From The REV. J. EDGAR, Lecturer on Geometrical Drawing, King's College.*

"I think your book to be a useful elementary one. The simplicity of the Problems in Part II., and the great help you have given in their solution, is calculated to assist those who find much difficulty in comprehending the subject.

"Young men reading for the army examinations will find assistance from Part I.; the Chapter on Scales will be of especial use to them. I am glad to see that you have given so much space to "Isometric Projection," and that you have added at the end of your work some hints on Shading and Colouring."

---

*From J. H. HAMMERSLEY, Esq., Head Master, School of Art, Bristol.*

"I was just about to write to you touching your book. I have not read it continuously in consequence, first, of excess of occupation, and again because my health has been so wretched of late. So far, however, I may state quite honestly and freely, that I never saw a recondite subject treated in a more lucid manner, and I am certain that no better work could be put in the hands of those students and

workmen to whom the subject peculiarly belongs. The book is at once learned and simple, which sums up the warmest judgment I can possibly entertain of it."

---

*From J. A. PHASE, Esq., Professor of Geometrical Drawing.*

"I have read Part II. of your work on Geometrical Drawing. I am sure of its being found of great service in obtaining an acquaintance with the principles of that subject, for whatever purpose such acquaintance may be needed. With regard to the special requirements for the present Military Competitive Examinations much useful information is contained in a small space."

---

*From W. H. SOUNES, Esq., Head Master, School of Art, Sheffield.*

"I thank you for your book on Geometrical Drawing, Part II. I have carefully examined it, and am convinced it will be found the most useful practical book yet published on the subjects of which it treats. I shall be glad to see the First Part, and to recommend both to all students in the Mechanical Class in this School."

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*From E. J. OLDFIELD, Esq., Professor of Geometrical and Military Drawing, from the Royal Academy, Brussels.*

"I have now had time to examine your book, with the attention it deserves, and have much pleasure in assuring you that I consider it by far the most *useful work* upon the subject hitherto published. I have compared it with most of the Continental and English publications, and find it a great deal more *practical*, owing, in a great measure, to the judicious and valuable collection of examples for practice, which examples really have some reference to the problems in the earlier part of the book; that is not the case in most of the similar works. The notes appended to the examples, indicating the problem to be employed, is a great improvement."

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*From W. BUSBRIDGE, Esq., Science Teacher, Woolwich.*

"Having carefully perused the enlarged edition of your excellent work on Geometrical Drawing, I have concluded that it is by far the best and most complete work on the subject in our language for students and all those who are desirous to obtain a knowledge of this important branch of education.

"I have just forwarded an order to the publishers for 25 copies."

## PRACTICAL GEOMETRY.

### DEFINITIONS.

(1) A *solid* is a body which has length, breadth, and thickness.

The elements of a solid are points, lines, and surfaces.

(2) A *point* has position but no magnitude, *e.g.*, the position of the intersection of two lines.

(3) A *line* has length only. Lines are straight or curved, as A and B.



(4) *Surfaces* are flat or plane, and curved.

(5) *Parallel lines* are lines situated in the same plane and equally distant from each other throughout, *e.g.*, the opposite edges of a book.

(6) A *horizontal line* is level, as A.

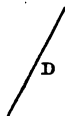
It takes its name from the horizon, to which it is parallel.

(7) A *vertical line* is a line perpendicular to the horizon, as

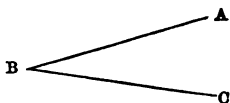


If a weight be suspended by a thread in a still atmosphere, the thread will represent a vertical line.

(8) *Oblique* or *inclined lines* are straight lines which are neither horizontal nor vertical, as



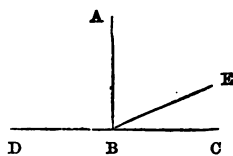
(9) A *plane rectilineal angle* is the inclination of two straight lines to one another, meeting or cutting each other in a point, as



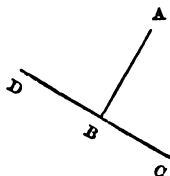
(10) "When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a right angle; and the straight line which stands on the other is called a perpendicular to it."

The angles  $A B C$ ,  $A B D$ , are right angles.

*Obs.* 1. Because  $D B$  is in the same straight line with  $B C$ , the angles  $A B C$ ,  $A B D$  are called *adjacent* angles, while the angles  $A B D$ ,  $A B E$  are called *contiguous* angles,  $B E$  not being in the same straight line with  $D B$ .



*Obs.* 2. A perpendicular line need not be a *vertical* line. The angles  $A B C$ ,  $A B D$  are right angles, and  $A B$  is perpendicular to  $D C$ , as in the preceding figure.



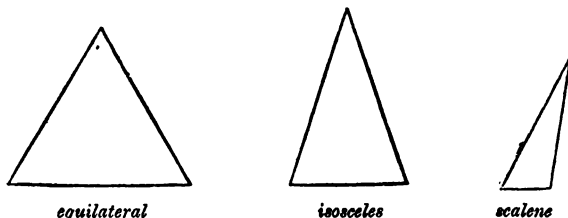
(11) An *acute* angle is less than a right angle, as  $E B C$  in last figure but one.

(12) An *obtuse* angle is greater than a right angle, as  $E B D$ .

*Obs.* An angle is named by a letter at the apex, as  $E$ , def. 9.

(13) A *rectilineal figure* is a plane figure contained by straight lines.

When contained by three straight lines, the figure is called a *triangle*.



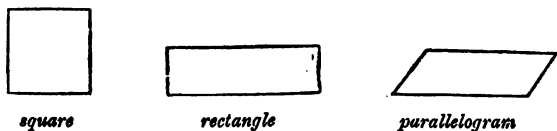
(14) An *equilateral* triangle has three equal sides.

(15) An *isosceles* triangle has two equal sides.

(16) A *scalene* triangle has three unequal sides.

(17) A rectilineal figure of four sides is called a *quadrilateral*.

(18) When the opposite sides of a quadrilateral are parallel, it is called a *parallelogram*. The three following figures are parallelograms.



(19) A *square* has all its sides equal, and all its angles right angles.

(20) A *rectangle* has all its angles right angles, but all its sides are not equal.

(21) A *rhombus* is a parallelogram having all its sides equal, but its angles are not right angles.

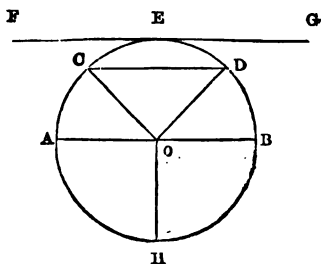
(22) A *rhomboid* is a parallelogram having only its opposite sides equal, but its angles are not right angles.

(23) Rectilineal figures of more than four sides are called *polygons*.

A *regular* polygon has all its sides and all its angles equal, otherwise it is termed *irregular*. A polygon takes its name from the number of sides of which it is composed.

A *pentagon* has five sides; a *hexagon*, six; a *heptagon*, seven; an *octagon*, eight; a *nonagon*, nine; a *decagon*, ten; an *undecagon*, eleven; and a *dodecagon*, twelve sides.

(24) A *circle* is a plane figure contained by one line called the circumference, every part of which is equally distant from a point within the circle, called the *centre*.



(25) A line drawn through the centre *o*, and terminated both ways by the circumference, is called a *diameter*, as *A B*.

(26) A line drawn from the centre to the circumference is called a *radius*, as *o C*, *o D*.

(27) The space enclosed by two radii is called a *sector*, as *C D O*.

(28) An *arc* is any part of the circumference of a circle, as *C E D*.

(29) The straight line joining the extremities of an arc is called a *chord*, as *C D*.

(30) That part of a circle contained by a chord, and the arc it cuts off is called a *segment*, as *C E D*.

That part of a circle contained by a diameter, and the part of the circumference it cuts off, is called a *semi-circle*, as *A E B*.

(31) A *quadrant* is a sector of  $90^\circ$ , whose radii are at right angles to each other, o h b.

(32) A line which touches a circle, or other curve, without cutting it, is called a *tangent*, as f g, touching the circle at e.

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#### DIRECTIONS.

In the study of Geometrical Drawing, the student must provide himself with a case of good instruments (those made of *white metal* are the best), which comprises the following :—

*Compasses.* One large pair having a moveable leg to insert a pencil or pen for drawing circles of long radii, and one small pair called *dividers*.

*Bow-pencil* and *Bow-pen* for drawing circles of short radii.

*Drawing pens.* Two ; one for fine, and one for thick lines.

*Protractor.* This instrument is used for laying down angles ; it also contains several useful scales.

A *Sector*.

*Marquois Scales*, comprising two rules and a triangle.

An explanation of the Protractor, Sector, and Marquois Scales, will be found in the problems in Practical Geometry.

Facility in the use of the instruments can only be acquired by practice. The Student will do well in beginning to ink in, to draw a series of lines, first *fine* and then gradually *thicker*, bearing in mind that a good line must be of the same thickness throughout. For inking in, always use Indian ink, which must be rubbed until it is quite black.

GENERAL RULES.

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1. Avoid drawing unnecessary lines, to obviate subsequent erasure.

2. Draw all lines from *left to right* and avoid pressing the pen against the edge of the triangle or other instrument, as it produces an uneven line.

3. Draw all the lines in pencil before beginning to ink in. For common use, a *HH* pencil is the best; for finer work, use a *HHH*. The pencil should be cut evenly to a fine point, if not, bad lines and inaccuracy will be the result.

4. In geometrical drawings dotted lines are employed as construction lines, and to show those parts of an object which are not seen in the view presented. They should be drawn of an equal length. An attention to this matter adds to the neatness and general appearance of the drawing.

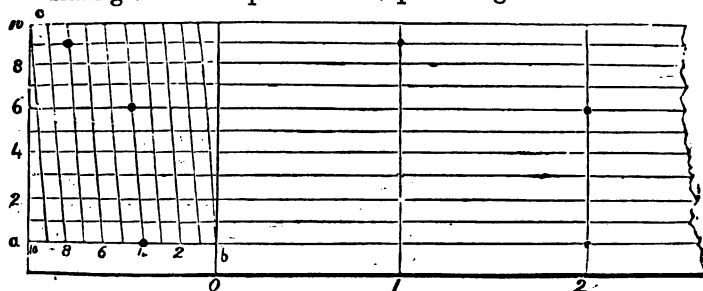
5. Before drawing a line, *try* the pen upon a *piece of waste paper*. Practice will show the necessity for this.

6. In using the compasses, bow-pen, or bow-pencil, press as lightly as possible to avoid puncturing the paper.

7. When an ordinary drawing is to be made, it will be sufficient to secure the paper to the drawing board by pins; but when a *finished* drawing is to be executed, the paper should be secured by gluing its edges to the board, first damping by means of a sponge, the whole surface of the paper.

*Obs.* As the solution of some of the problems in Practical

Geometry requires a knowledge of decimal notation, we shall give here a specimen of a plain diagonal scale.



Suppose it is required to take off 2·4 inches.

It will be observed that the first inch ( $a\ b$ ) is divided into 10 equal parts, each of which will be  $\frac{1}{10}$  of an inch, or  $\cdot 1$ ; two parts will be  $\frac{2}{10}$  of an inch, or  $\cdot 2$ , etc. Therefore, if we place one leg of the dividers at the point shown by the dot on the perpendicular drawn through 2, and the other leg at 4 in  $a\ b$ ; the distance included between these two points will be 2·4 inches.

Let it be required to take off 2·46 inches. This is shown by the second dots. Place one leg of the dividers at the point where the horizontal drawn through 6, in  $a\ c$ , intersects the diagonal drawn through 4, in  $a\ b$ , and the other leg at the point where the same horizontal intersects the perpendicular drawn through 2.

Let it be required to take off 1·79 inches. This is shown by the third dots. Place one leg of the dividers at the point where the horizontal drawn through 9, in  $a\ c$ , intersects the diagonal drawn through 7, in  $a\ b$ , and the other leg at the point where the same horizontal intersects the perpendicular drawn through 1.

*Note.* For the construction of a Diagonal Scale, see chapter on Scales.

## PROBLEMS.

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*Note.* In the figures on Practical Geometry, the student is requested to attend to the following:—Lines given are drawn *thin*, construction lines, *dotted*, and resulting lines, *thick*. The position of a point is shown by a small circle of which it is the centre.

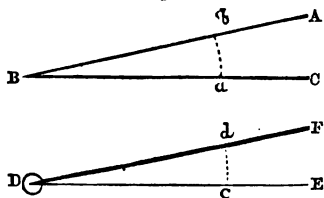
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## PROBLEM 1.

*At a given point in a straight line, to make an angle equal to a given angle.*

Let  $\angle ABC$  be the given angle, and  $DE$  the given line.

Fig. 1.

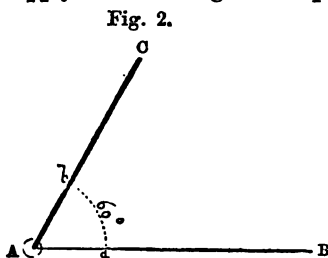


With centre  $B$ , and any convenient radius, describe an arc cutting  $AB$ ,  $BC$  in  $a$ ,  $b$ , and with centre  $D$  and same radius, describe an arc cutting  $DE$  in  $c$ . Set off upon this arc,  $cd$  equal to  $ab$ ;

join  $Dd$  and produce it to  $F$ , then will the angle  $EDF$  be equal to  $\angle ABC$ .

Let it be required to construct an angle equal to a given number of degrees, say  $60^\circ$ .

Apply the lower edge of the protractor (see explanation of protractor) to the given line  $AB$ , so that the centre mark be on  $A$ ; then make a mark against the upper edge at the line indicating  $60^\circ$ , and join this point to  $A$ .

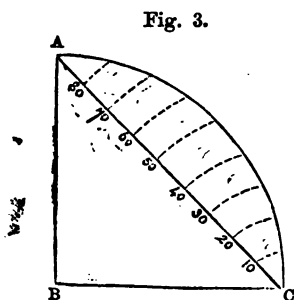


The line  $AC$ , thus found, will make with  $AB$  the required angle. If the given line were long enough, the angle may be constructed thus:—The protractor being applied as before, move it round, keeping the centre point upon  $A$ , until the line marked  $60^\circ$  coincides with  $AB$ . Draw  $AC$  along the edge, and the angle  $BAC$  is the angle required.

The operation in both cases, is simply that of transferring the angle from the scale to the paper.

To construct a scale of chords to set off angles, or to measure angles already laid down.

Describe the quadrant  $ABC$ , and join  $AC$ ;  $AC$  is a chord of  $90^\circ$ . Divide the arc  $AC$  into nine equal parts, and transfer the points of division upon the chord  $AC$ , each of which will be a chord of  $10^\circ$ . Sub-divide each of the divisions in  $AC$  into ten equal parts, and we shall have a scale of chords. Each of the sub-divisions in  $AC$  will be one degree, and if the circle were sufficiently large, these could again be sub-divided for minutes.

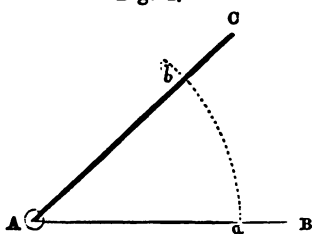


Each of the sub-divisions in  $AC$  will be one degree, and if the circle were sufficiently large, these could again be sub-divided for minutes.

To construct the angle  $BAC$  (Fig 2), by means of a scale of chords. The chord of  $60^\circ$  is equal to radius (Euc. iv. prop. 15 Cor.). Therefore, with centre  $A$  and any radius, describe an arc, cutting  $AB$  in  $a$ , and set off from  $a$ ,  $ab$  equal to  $Aa$ . Join  $ab$  and produce it, and  $BAC$  will be an angle of  $60^\circ$ .

Let it be required to draw a line from  $A$ , that shall make with  $AB$  an angle of  $40^\circ$  (Fig. 4). Describe an arc with a radius equal to chord of  $60^\circ$ , cutting  $AB$  in  $a$ , and from  $a$ , set off  $ab$  equal to  $40^\circ$  taken from the scale of chords. Join  $Ab$  and produce it to  $c$ ;  $BAC$  is the angle required.

Fig. 4.

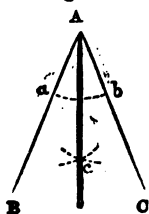


To measure the angle  $BAC$ , describe from centre  $A$ , the arc  $ab$  with radius equal to chord of  $60^\circ$ , then take the distance  $ab$  in the compasses, apply it to the scale of chords, and, placing one leg at zero the other will extend to the number indicating the degrees in the given angle.

## PROBLEM 2.

*To bisect a given rectilineal angle.*

Fig. 5.

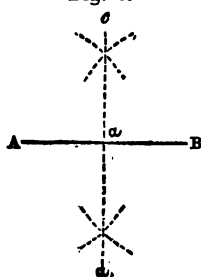


Let  $BAC$  be the given angle. With centre  $A$  describe the arc  $ab$ , cutting  $AB$ ,  $AC$ , in  $a$ ,  $b$ ; and with centres  $a$ ,  $b$ , and radius  $Aa$ ,  $Ab$ , describe arcs intersecting in  $c$ . Join  $Ac$ , and the angle  $BAC$  is bisected.

## PROBLEM 3.

*To bisect a given straight line.*

Fig. 6.

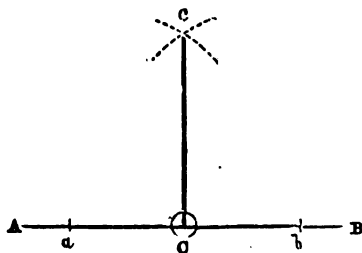


Let  $AB$  be the given line. From centres  $A, B$ , and with a convenient radius, describe arcs intersecting in  $c, d$ . The line joining  $c, d$ , bisects  $AB$  in  $a$ .

## PROBLEM 4.

*At a given point in a given straight line to erect a perpendicular.*

Fig. 7.

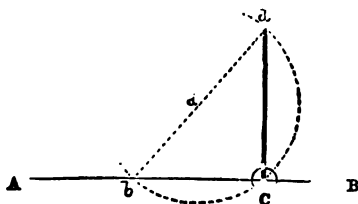


Let  $AB$  be the given line, and  $c$  the given point near the middle of  $AB$ . From  $c$ , set off equal distances  $ca, cb$ . With  $a, b$ , as centres, describe arcs intersecting in  $e$ , and join  $ec$ ;  $ec$  is perpendicular to  $AB$ .

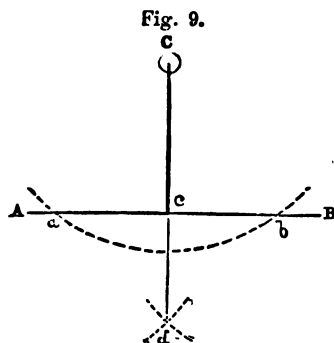
*Second case.* Let the given point  $c$  be near one end of the

line  $AB$ . Take any point  $a$  above  $AB$ , and with  $a$  as a centre, and radius  $ac$ , describe a circle cutting  $AB$  in  $b$ . Join  $ab$ , and produce it to meet the circle in  $d$ . The line joining  $d, c$ , is perpendicular to  $AB$ .

Fig. 8.



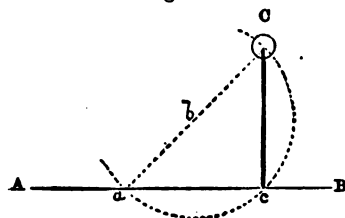
*Third case.* To draw a perpendicular to a given line from



a point without. Let  $c$  be the given point, and nearly opposite the middle of  $AB$ . From  $c$  as a centre and with a convenient radius, describe an arc cutting  $AB$  in  $a, b$ . With  $a, b$  as centres, describe arcs intersecting in  $d$ . Join  $c, d$ , and  $cd$  is the perpendicular required.

*Fourth case.* Let  $c$  be nearly opposite the end of  $AB$ .

Fig. 10.



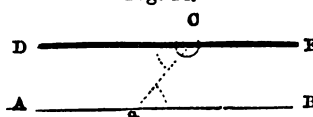
In  $AB$ , take any point  $a$ , and join  $ac$ . Bisect  $ac$  in  $b$  (Prob. 3), and with centre  $b$  and radius  $ba$  or  $bc$ , describe a semi-circle cutting  $AB$  in  $e$ . The line joining  $ce$  is perpendicular to  $AB$ .

### PROBLEM 5.

*Through a given point to draw a line parallel to a given straight line.*

Let  $AB$  be the given line, and  $c$  the given point. In  $AB$ ,

Fig. 11.



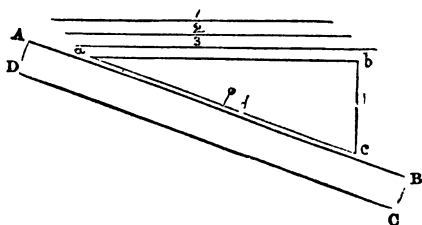
take any point  $a$ , and join  $ac$ . At  $c$  in  $ac$ , make the angle  $acd$  equal to  $cab$  (Prob. 1), and produce  $dc$  to  $e$ ;  $de$  is parallel to  $AB$ .

Problems 4 and 5 can be solved practically by means of the Marquois Scales, which we shall now proceed to describe.

### MARQUOIS SCALES.

These Scales consist of two rectangular rules, and a right

Fig. 12.



angled triangle.

$\triangle ABC$  represents one of the rules, and  $abc$  the triangle, whose hypotenuse or longest side  $ac$ , is three times  $bc$ , its

shortest side. Near the centre of  $ac$  is an index,  $d$ .

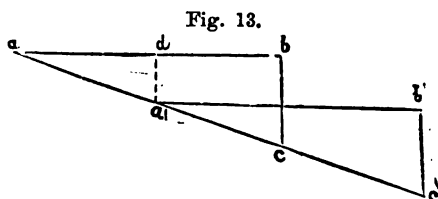
If the line marked 1 be given, and it were required to draw a number of lines parallel to it, we should proceed thus:—Make  $ab$  coincide with the given line, then apply the rule to the hypotenuse  $ac$ , and holding it firm with the left hand, move the triangle along the edge  $AB$ , and draw the lines 2, 3, etc.

If it were required to draw a line perpendicular to a given line from a point *in* or *without* the line, make the shortest side  $bc$  coincide with the given line, then apply the edge  $AB$  of the rule along  $bc$ , and holding the rule firm with the left hand, move the triangle until  $ab$  be sufficiently near the given point as to admit the pencil passing through it, when drawing the line.

Now, besides being able to draw one line parallel to

another, we can also draw it at any required distance; and this is one property of the Marquois Scales which makes them so valuable to the military student. When a number of lines have to be drawn parallel to each other, as in the slopes of a work, much time is saved, and greater accuracy attained, by the use of the Scales, than by setting off the distances with the compasses.

Let  $abc$ , Fig. 13, be the original position of the triangle.



Apply the rule along the hypotenuse  $ac$ , and, holding it as before directed, move the triangle to

the position  $a'b'c'$ , draw  $a'd$  perpendicular to  $ab$ , and we have, by similar triangles,  $a'b'c'$ ,  $a'da'$

$$\begin{aligned} a'd : a'a :: c'b' : c'a' \\ :: 1 : 3 \end{aligned}$$

Now, while  $a$  has been moved from  $a$  to  $a'$ , it has also been moved in a direction perpendicular to  $a'b'$  a distance equal to  $a'd$ ; and, if  $a'a'$  be considered to represent 15 yards, a line  $a'b'$  has been drawn parallel to  $ab$ , at a distance from it,  $= a'd = \frac{1}{3}$  of  $a'a = 5$  yards.

This principle is embodied in the construction of the Marquois scales, each of which consists of two parts, an outer or *artificial* scale, and an inner or *natural* scale. The latter is the scale to which the drawing is made, and the number written below it, signifies the number of units whether of yards or chains, etc., represented by one inch. Take for example the scale of 30 (the 30 being written under the natural scale and below the zero of the artificial scale), it

will be observed, that the primary divisions are marked from left to right, and numbered 1, 2, 8, etc., the first being subdivided into 10 parts. Taking the sub-divisions as single units of measure, three primary divisions will contain 30, and the scale will be called a scale of 30 units to one inch.

The primary divisions upon the artificial scale are marked from zero both ways, and numbered 10, 20, 30, etc. The divisions on the artificial scale are three times those on the natural scale, thus giving the scales the same ratio that the longest side of the triangle has to the shortest side.

Let it be required to draw a line parallel to a given line at a distance of 20 yards from it.

Apply the bevelled edge of the triangle to the given line; then apply the rule to the longest side of the triangle, so that the zero on the artificial scale, is against the index of the triangle; hold the rule firm with the left hand, and sliding the triangle, either to the right or left, until its index is against 20, draw the line. If, in drawing the line, we had used the scale of 30, then a line would have been drawn parallel to a given line at a distance of 20 yards from it, scale 30 yards to one inch.

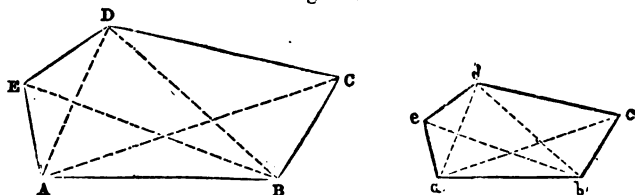
It may be as well to remark, that the sub-divisions in the first primary division, on the natural scale, may represent any number of units. For example, taking each of the sub-divisions on the scale of 30 as 10 units, each primary division will contain 100, and we shall thus have a scale of 300 to one inch. In this way, the scales may be varied to any extent.

Since, in using the scales, we work from the artificial scale, any error which may be made in adjusting the instruments, will only be one-third in the drawing.

## PROBLEM 6.

*To construct a rectilinear figure similar to a given rectilinear figure, but less in area.*

Fig. 14.



There are several ways of solving this problem, but we shall confine ourselves to two, based upon Probs. 1 and 5.

Let  $A B C D E$  be the given figure, and  $a b$  a given line upon which to construct the required figure.

Draw the diagonals  $A D$ ,  $A C$ ,  $B D$ ,  $B E$ ; then from  $a$ ,  $b$ , which must be in the same straight line with or parallel to  $A B$ , with the Marquois scales, draw indefinite lines parallel to these, and again, draw lines respectively parallel to  $A E$ ,  $E D$ ,  $D C$ ,  $C B$ . The points of intersection of these lines, with those first drawn, determine the Fig.  $a b c d e$ .

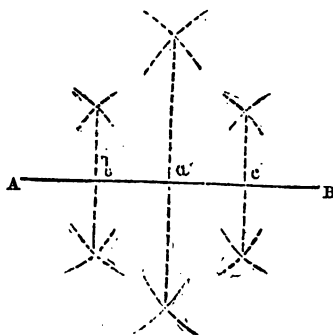
*Second Solution.* Having drawn the diagonals of the given figure, take the line  $a b$  and make the angle  $e a b$  equal to  $c A B$  by Prob. 1. Again, make the angle  $a b c$  equal to  $A B C$ , and we determine the side  $b c$ . By proceeding in the same manner, we determine the points  $d$ ,  $e$ .

*Note.* This problem has a practical application, in Orthographic Projection, in making drawings of the same object to different scales; for  $a b c d e$  is a copy of  $A B C D E$  on a reduced scale.

## PROBLEM 7.

*To divide a given straight line into any number of equal parts.*

Fig. 15.

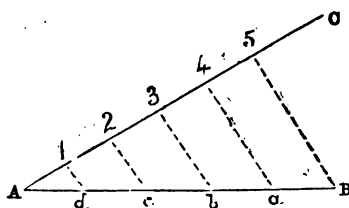


First, when an even number of parts is required, say four. Let  $AB$  be the given line. Bisect  $AB$  in  $a$  (Prob. 3). Again, bisect  $aA$ ,  $aB$ , in  $b$ ,  $c$ , and  $AB$  will be divided into four equal parts.

The line may be divided into eight equal parts, or any multiple of four by repeated bisection.

*Second case.* Let it be required to divide  $AB$  into an

Fig. 16.



uneven number of parts, say five. From  $A$ , draw  $AC$ , making any convenient angle with  $AB$ , and upon  $AC$ , set off any equal distances  $A1$ ,  $12$ , etc. Join  $B5$ , and through points  $4$ ,  $3$ ,  $2$ ,  $1$ ,

draw lines parallel to  $B5$ , cutting  $AB$  in  $a$ ,  $b$ ,  $c$ ,  $d$ ;  $AB$  is divided into five equal parts in the points  $a$ ,  $b$ ,  $c$ ,  $d$ .

It is found convenient, in practice, to divide lines by trials, which we shall briefly notice. Let it be required, for example, to divide the line  $AB$  (Fig. 16) in this manner. Open the compasses to the apparent extent

of  $\frac{1}{2}$  of the given line, and step it along A B. If it be found too great, or too little, regulate the compasses, and repeat the trial until the exact distance is obtained.

If the number of parts into which the line is to be divided can be resolved into factors, we may proceed thus:—Taking 9 as the number, first divide the line into three parts, and these again into three parts, and the whole line will be divided into nine parts. Taking 12 as the number, first bisect the whole line, then bisect each half, and lastly, sub-divide each fourth part into three parts.

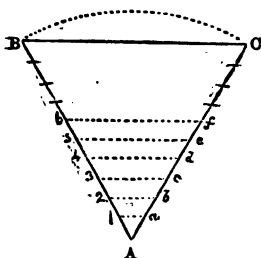
*Obs.* In thus dividing lines, care must be taken, when making a finished drawing, to avoid puncturing the paper. Lines may also be divided by means of the Sector, which we shall here briefly describe.

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## THE SECTOR.

This instrument which derives its name from Eu. book III. def. 10, is of universal application. It consists of two equal rulers, called legs or limbs. These legs—representing two radii of a circle—are moveable about a joint, the centre of which is the centre of the circle. The sectoral lines proceed from this centre in pairs, one line upon each leg. Upon one face are the *line of lines* marked L, *lines of chords* marked C, *lines of secants* marked S, and *lines of polygons* marked P O L. Upon the other face are *lines of sines* marked S, and *lines of tangents* marked T.

Fig. 17.



Let  $AB$ ,  $AC$ , be two sectoral lines. Set off upon  $AB$  any number of equal parts 1, 2, 3, 4, 5, 6, and, from these points, draw lines parallel to  $BC$ , cutting  $AC$  in  $a$ ,  $b$ ,  $c$ , etc.

Now, the distances  $1a$ ,  $2b$ ,  $3c$ , etc., have, to each other, the same ratio as the numbers 1, 2, 3. For example, the distance  $2b$  has the same ratio to  $5c$ , as the number 2 has to the number 5, or  $\frac{2}{5}$ .

If the equal divisions be continued to  $B$ ,  $C$ , making 10 at these points, and if  $BC$  measure 1.1 inch; then  $1a$  will be  $\frac{1}{10}$ ,  $2b$  will be  $\frac{2}{10}$  or  $\frac{1}{5}$  of a line 1.1 inch long. In this way, we can find any required part of a given line.

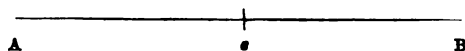
These operations are performed by means of the sector, thus:—To find  $\frac{2}{5}$  of a line 4.4 inches long, apply the given line, *i.e.*, make it a transverse distance (as the lines  $1a$ ,  $2b$ , etc., Fig. 17) between the numbers 9 and 9 on the line of lines; then the transverse distance between 4 and 4 will be the length required. In applying the given line, and in taking off the required length, the innermost line must be used as it is the sectoral line, and, therefore, the only line of the three which goes to the centre. To divide a line 4.4 inches into 9 equal parts, apply the line as before, then take off the transverse distance between 1 and 1, and step it along the given line 9 times.

Owing to imperfections in the instrument, and to want of care in using it, little errors will occur in practice, which will require correction.

## APPLICATION OF THE LINE OF LINES.

1. To bisect a given line  $A B$ , 2·3 inches long.

Fig. 18.

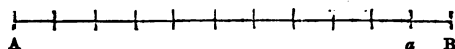


Make the given line a transverse distance between 10 and 10, then take the transverse distance between 5 and 5, and set it off from  $A$  to  $c$ ; the line  $A B$  is bisected in  $c$ .

2. To divide the line  $A B$  in the last example into eleven equal parts.

Since the divisions on the line of lines only extend to 10, we must take a sub-multiple of the number of parts into which the line is to be divided, and make the given line a transverse distance between the numbers corresponding to this sub-multiple. Make, therefore,  $A B$  a transverse

Fig. 19.



distance between 5·5 and 5·5, which is a sub-multiple of 11; then take the transverse distance between 5 and 5, and set it off from  $A$  to  $a$ . Take  $a B$  and step it along  $A B$ , and the line will be divided into eleven equal parts.

In making  $A B$ , which is to contain eleven divisions, a transverse distance between 5·5 and 5·5, the numbers on the sector are supposed to be changed, thus, ·5 becomes 1; 5 becomes 10, and 10 becomes 20. In dividing  $A B$  into eleven equal parts, instead of taking the transverse distance between 5 and 5 and setting it off from  $A$  to  $a$ , we might have taken the transverse distance between ·5 and ·5. The result, however, will not be so accurate as in the

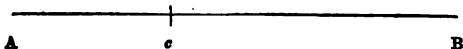
former case, owing to the indistinctness of the numbers near the centre of the sectoral lines. The student is, therefore, recommended to employ the method we have given.

It will be observed that the divisions upon the line of lines agree with the decimal notation, each primary division being divided into 10 sub-divisions, and according to the value attached to each of these sub-divisions, so 10 may stand for 100, 1000, etc., and, thus, a line may be divided into 50 or 500 parts, as well as into 5 parts.

3. To determine  $\frac{2}{7}$  of a line 3 inches long.

As the denominator of the fraction exceeds 10, the distance cannot be taken from the primary divisions, and we must, therefore, have recourse to the decimal notation. Multiply the numerator and denominator of the fraction by 3. We thus get  $\frac{6}{21}$ . Take the given line A B in the com-

Fig. 20.

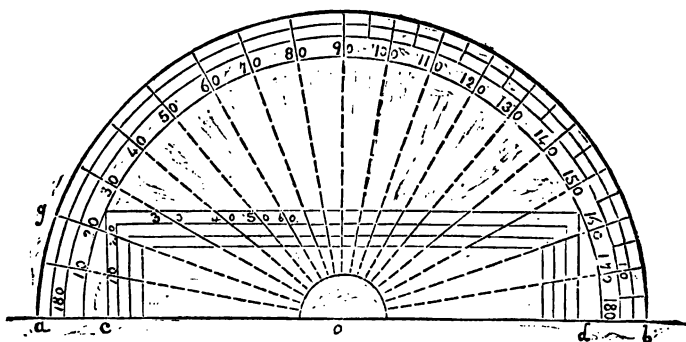


passes, and make it a transverse distance between 8·1 and 8·1; then, the transverse distance between 2·7 and 2·7, set off from A to c, will contain the number of parts required. The object, in multiplying by 3, is to bring the measurement near the end of the line, for the reason previously given.

The problem may be verified thus,  $\frac{2}{7} = \frac{1}{3}$ ; therefore, the number of parts required will measure  $\frac{1}{3}$  of A B. Now, A c is  $\frac{1}{3}$  of A B.

Further explanation of the sector will be given when discussing those problems, whose solutions can be effected by its aid.

*Obs.* The division of a line into any given number of parts has a practical application in the construction of scales.



### THE PROTRACTOR.

This instrument is used to lay down angles. It is sometimes made of brass of a semi-circular form, and sometimes of ivory of a rectangular form, as shown in the figure.

If the circumference of a circle be divided into 360 equal parts, each part is called one degree (written  $1^\circ$ ), and if from these points of division, lines be drawn to the centre of the circle, the opening or angle between any two consecutive lines, will be  $1^\circ$ . Again, the angle between the lines including 15 divisions, will be an angle of  $15^\circ$ , and so on. In the figure, the semi-circle is divided into 18 equal parts (each part containing  $10^\circ$ ), and is numbered from left to right, from zero to 180, and in the same manner from right to left. In the centre of the protractor there is a mark or index, *o*. The manner of using the instrument is this:—Suppose it were required from a given point in a line, *AB*, to draw a line making with it an angle of  $20^\circ$ . Make *ab* or *cd* coincide with the given line, so that the index *o* be upon the given point, then make a mark on the paper at *g*, or the edge-

of the protractor on the line passing through 20. The line, drawn from  $g$  to  $o$ , will make with  $AB$  an angle of  $20^\circ$ .

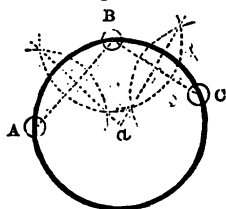
The construction of both instruments must be sufficiently evident, from the drawing, without further explanation.

### PROBLEM 8.

*To describe a circle which shall pass through three given points not in the same straight line.*

Let  $A, B, C$ , be the given points. Join  $AB, BC$ , and bisect them (Prob. 3). From  $a$ , the point of intersection of the bisecting lines, and with any of the given points as a radius, describe a circle; this circle will pass through the given points  $A, B, C$ .

Fig. 21.

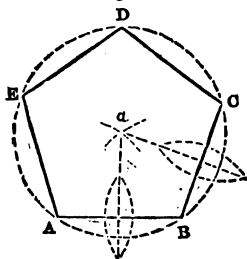


### PROBLEM 9.

*To describe a regular polygon on a given line.*

Let  $AB$  be the given line, and let the required polygon be a pentagon. From Euc. I., 32, Cor. 1, "All the angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides." From this corollary, we can deduce a formula for finding the angle of any polygon. Let  $x$  equal the

Fig. 22.



number of degrees in the required angle, and  $y$  the number of sides of the given polygon.

$$\text{Then } \frac{2n-4}{y} \times 90 = x$$

Substituting the value of  $y$  in the case of a pentagon,  
 $\frac{10-4}{5} \times \frac{18}{90} = x = 6 \times 18 = 108^\circ$

The angle of a polygon may be also found thus :—Take the case of a hexagon. Divide  $360^\circ$ , the number of degrees in a circle, by 6, the number of sides of the given polygon; the quotient is  $60^\circ$ , which gives the angle at the centre of the polygon. Subtract  $60^\circ$  from  $180^\circ$ , the number of degrees in a semi-circle, and we obtain  $120^\circ$ , the measure of the angle required.

Take the case of a pentagon :—Angle of the figure =  
 $180^\circ - \frac{360^\circ}{5} = 108^\circ$ .

*Construction.* At the point  $B$  in  $AB$ , make the angle  $ABC$  equal  $108^\circ$ , and make  $BC$  equal to  $AB$ . Through the points  $A, B, C$ , describe a circle by the last problem. Take the distance  $AB$  or  $BC$  in the compasses, and from  $A, C$ , set off the points  $D, E$ . Join  $CD, DE, EA$ , and  $ABCDE$  will be a pentagon.

*Second case.* The figure may be constructed by means of the sector, thus :—Make  $AB$  a transverse distance between 5 and 5 (between 6 and 6 for a hexagon, 7 and 7 for a heptagon, and so on), on the line of polygons marked  $POL$ ; then the transverse distance between 6 and 6 will give the radius of the circumscribing circle (Euc. iv., 15, Cor.)

With this distance, and with  $A, B$ , as centres, describe arcs intersecting in  $a$ . With centre  $a$ . and radius  $aA$ ,

or  $a$   $B$ , describe a circle, and step along  $B$   $C$ ,  $C$   $D$ , etc., each equal to  $A$   $B$ .

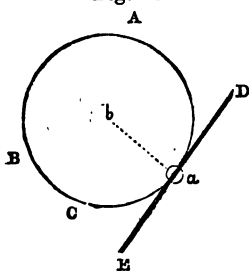
To inscribe a polygon in a given circle, make the radius a transverse distance between 6 and 6; then the transverse distance between 5 and 5 will be the side of an inscribed pentagon. Take this distance and step it along the circumference of the circle 5 times. From Euclid's corollary, given above, an inscribed hexagon is obtained by stepping the radius along the circumference 6 times.

The present problem, which has a practical application in the study of fortification in determining the figure upon which a fortress is supposed to be constructed, enables us, as in the first case, to describe polygons of any number of sides; and, in the second case, any number of sides from 4 to 12, that being the highest number on the line of polygons.

#### PROBLEM 10.

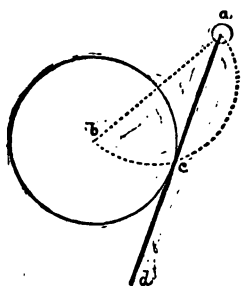
*At a given point in the circumference of a circle, to draw a straight line a tangent to the circle.*

Fig. 23.



Let  $A B C$  be the given circle, and  $a$  the given point in the circumference. Join  $a$  to  $b$ , the centre of the circle, and, through  $a$ , draw  $E D$  at right angles to  $a b$ ;  $E D$  is a tangent to the circle  $A B C$  at the point  $a$ .

Fig. 24.

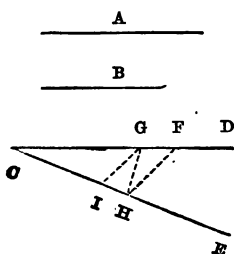


*Second case.* To draw a straight line a tangent to a circle from a point *without*. Let *a* be the given point. Join *a b*, and upon it describe a semi-circle, cutting the given circle in *c*. Join *a c*, and produce it to *d*; *a d* is the tangent required.

## PROBLEM 11.

*To find a third proportional to two given lines.*

Fig. 25.



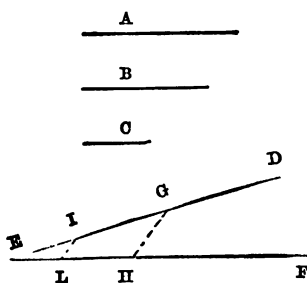
Let *A*, *B*, be the given lines.

Draw *D C*, *C E*, making any angle. Make *C F* equal to *A*, and *C G*, *C H*, equal to *B*. Join *F H*, and, through *G*, draw *G I* parallel to *F H*. Then *G O I*, *F C H*, are similar triangles; and *C F* is to *C H*, as *C G* is to *C I*, i.e., *C I* is the third proportional to *A*, *B*.

## PROBLEM 12.

*To find a fourth proportional to three given lines.*

Fig. 26.



Let *A*, *B*, *C*, be the given lines. Draw *E D*, *E F*, as in the last problem. Make *E G*, *E H*, equal to *A*, *B*. Join *G H*. Again, make *E I* equal to *C*, and draw *I L* parallel to *G H*. Then *E H* is to *E G*, as *E L* is to *E I*, i.e., *E L* is the fourth proportional to *A*, *B*, *C*.

The two preceding problems can be solved by means of the sector, thus:—Make *A* (Fig. 26), a transverse distance between 5 and 5 on the line of lines; then take *B* in the compasses, and find between what divisions or sub-divisions it is a transverse distance.

It will be found a transverse distance between 4 and 4. Next make *C* a transverse distance between 5 and 5, and the transverse distance between 4 and 4 will be the fourth proportional required.

To find a third proportional (Fig. 25). Take a third line equal to the second, and find a fourth proportional to the three, thus:—1° Make *A* a transverse distance between 5 and 5. 2° Make *B* a transverse distance as was done in the last case. 3° Make *B* also a transverse distance between 5 and 5, and then the transverse distance 2° will be the third proportional required.

In both problems, it will be observed that we have made the first line a transverse distance between 5 and 5. The same result may be obtained by using other numbers on the line of lines. The length of the given lines will determine which number should be selected.

The following is another method for finding the fourth proportional to three given lines.

“Set off from the centre (on the line of lines) a lateral distance equal to the first term, and open the sector till the transverse distance at the division thus found, expressing the first term, is equal to the second term; again, extend to a point whose lateral distance from the centre is equal to the third term, and the transverse distance at this point will be the fourth term required.

“If the legs of the sector will not open far enough to make the lateral distance of the second term, a transverse distance

at the division expressing the first term, take any aliquot part of the second term, which can be conveniently made such transverse distance, and the transverse distance at the third term will be the same aliquot part of the fourth proportional required.

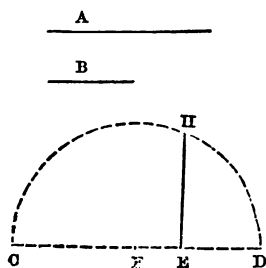
"A third proportional to two given lines is found by taking a third line equal to the second, and finding the fourth proportional to the three lines."

### PROBLEM 13.

*To find a mean proportional between two given lines.*

Let A, B, be the given lines. Make CD equal to the sum

Fig. 27.



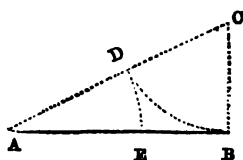
of A and B. On CD make CE equal to A. Bisect CD in F (Prob. 3), and with centre F and radius FC or FD, describe a semi-circle. From E, draw EH, at right angles to CD, and cutting the circumference in H; EH is a mean proportional between A and B, i.e., A is to EH, as EH is to B, or numerically thus:—If A equal 9 and B equal 4, then  $\sqrt{9 \times 4} = 6 = EH$ , or  $9 : 6 :: 6 : 4$ . Hence the mean proportional between any two numbers is the square root of their product.

### PROBLEM 14.

*To divide a straight line into extreme and mean ratio, i.e., into two unequal parts, so that the whole line shall have to the greater part the same ratio, that the greater part has to the less.*

Let  $AB$  be the given line. From  $B$ , draw the perpendicular  $BC$ , equal to half  $AB$ , and join  $AC$ .

Fig. 28.

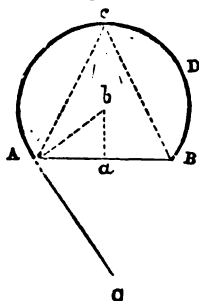


With centre  $C$ , and radius  $CB$ , describe an arc, cutting  $AC$  in  $D$ ; and with centre  $A$ , and radius  $AD$ , describe an arc, cutting  $AB$  in  $E$ . The line  $AB$  is divided in  $E$  into extreme and mean ratio, *i.e.*,  $AD$  is to  $AE$ , as  $AE$  is to  $EB$ .

## PROBLEM 15.

*On a given straight line, to describe a segment of a circle which shall contain an angle less than a right angle.*

Fig. 29.



Let  $AB$  be the given straight line. Make  $BAC$  equal to the given angle.

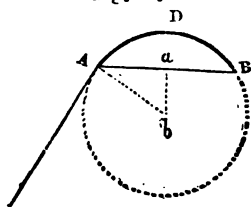
Bisect  $AB$  in  $a$  (Prob. 3), and from  $A$ ,  $a$ , erect perpendiculars\* to  $AC$ ,  $AB$ , intersecting in  $b$ . From centre  $b$ , and radius  $ba$  or  $bb$  describe a circle; the segment  $ABD$  will contain an angle equal to  $BAC$ . In the segment  $ABD$ , take any point  $c$ , and join  $cA$ ,  $cB$ ; then the angle  $ACB$  is equal to  $BAC$ .

*Second case.* Let the given angle be greater than a right angle.

---

\* When a line is to be drawn perpendicular to a given line, the student is recommended, in all cases, to employ the Marquois scales and triangle.

Fig. 30.



Let  $AB$  be the given line. Make the angle  $BAC$  equal to the given angle. Bisect  $AB$  in  $a$ , and from  $a$ ,  $A$ , erect perpendiculars to  $AB$ ,  $AC$ , intersecting in  $b$ . From centre  $b$ , and radius  $bA$ , or  $bB$ , describe a circle; the segment  $ABD$  will contain the given angle.

If it were required to describe a segment of a circle to contain a right angle, we should bisect  $AB$  in  $a$ , and upon it describe a semi-circle; then take any point in the circumference, and join it to  $A$ ,  $B$ , and the angle contained by these two lines will be a right angle.

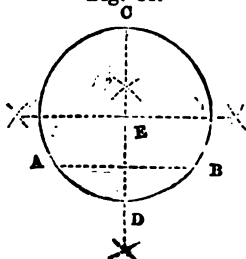
*Obs.* This problem enables us to cut off from a circle a segment, which shall contain a given angle.

Let  $ABD$  be the given circle (see Fig. 29). From any point  $A$ , in the circumference, draw a tangent to the circle as  $AC$  (Prob. 10). From  $A$ , draw a chord  $AB$ , making with  $AC$  an angle equal to the given angle;  $ABD$  will be the segment required.

#### PROBLEM 16.

*To find the centre of a given circle.*

Fig. 31.

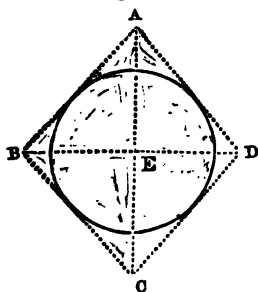


Let  $ABC$  be the given circle. Draw any chord  $AB$ , and bisect it by a perpendicular produced to meet the circumference in  $C$ ,  $D$ . Bisect  $CD$  in  $E$ ;  $E$  is the centre of the circle.

To find the centre of a circle by means of the Marquois scales and triangle only.

With the bevelled edge of the triangle, draw  $AD$ , touching the circle; and parallel to it draw  $BC$  also touching the circle. In the same manner, draw  $BA$ ,  $CD$ , cutting the former lines in  $A$ ,  $B$ ,  $C$ ,  $D$ . Join the diagonals  $AC$ ,  $BD$ ; their point of intersection  $E$  is the centre of the circle. The lines  $BA$ ,  $AD$ , may be drawn in any direction whatever. We have only to produce them till they meet, and draw  $BC$ ,  $CD$ , respectively, parallel to them.

Fig. 32.

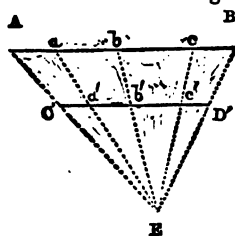


It will be observed (see Prob. 10) that this construction is not strictly geometrical.

# PROBLEM 17.

To divide a line proportionally to a given divided line.

Fig. 33.



Let  $AB$  and  $CD$  be the given lines,  $AB$  being divided into the unequal parts  $Aa$ ,  $ab$ ,  $bb$ ,  $bc$ ,  $cB$ .

Parallel to  $AB$ , draw  $a'd'$  equal to  $CD$ . Join  $Aa'$ ,  $Bb'$ , and produce them till they meet in  $E$ . Join each of the divisions in  $AB$  to  $E$ , and then will  $a'd'$  be divided in  $a'$ ,  $b'$ ,  $b'$ ,  $c'$ , proportionally to  $AB$ .

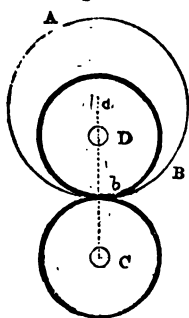
The problem may also be solved thus:—Let  $AB$  (Fig. 16) represent the divided line. From  $A$ , draw  $As$ , making any

angle with  $AB$ , and equal in length to the undivided line. Join  $B$  to  $5$ , and through the points in  $AB$  draw lines parallel to  $B5$ ; then will  $AB$  be divided proportionally to  $AB$ .

### PROBLEM 18.

*From a given point, to describe a circle which shall touch a given circle.*

Fig. 34.



Let  $AB$  be the given circle and  $c$  the given point. Join  $a$ , the centre of the given circle, to  $c$ , cutting the circumference in  $b$ . From centre  $c$ , and radius  $cb$ , describe a circle. This circle will touch the given circle at the point  $b$ .

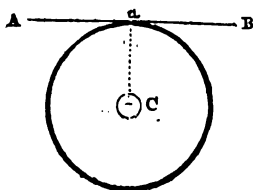
*Second case.* Let  $d$ , the given point, be within the given circle.

Join  $a$ ,  $d$ , and produce it to meet the circumference in  $b$ . With centre  $d$ , and radius  $db$ , describe a circle.

### PROBLEM 19.

*From a given point, to describe a circle which shall touch a given straight line.*

Fig. 35.



Let  $AB$  be the given straight line, and  $c$  the given point. From  $c$ , draw  $ca$  perpendicular to  $AB$ , Prob. 4. From centre  $c$ , and radius  $ca$ , describe a circle. This circle will touch the given line at the point  $a$ .

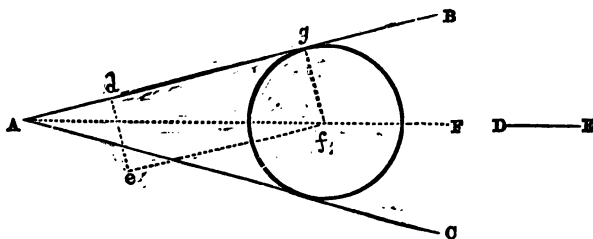
If it were required to draw a circle with a given radius touching a given straight line, at a given point, we should proceed thus :—Let  $AB$  be the given straight line (Fig. 35), and  $a$  the given point. From  $a$ , draw  $ac$ , perpendicular to  $AB$  (Prob. 4), making  $ac$  equal to the given radius. With centre  $c$ , and radius  $ca$ , describe a circle.

PROBLEM 20.

*To describe a circle, with a given radius, tangential to two lines containing a given angle.*

Let  $AB, AC$ , be the two lines containing the angle  $BAC$ ,

Fig. 36.

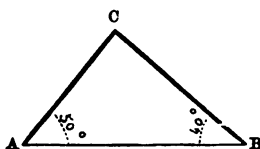


and let  $DE$  be the given radius of the circle. Bisect the angle  $BAC$  by  $AF$  (Prob. 2). The centre of the required circle will be in  $AF$ . In  $AB$ , take any point  $d$ . From  $d$ , draw  $de$  perpendicular to  $AB$ , and equal to  $DE$ . From  $e$ , draw  $ef$  parallel to  $AB$ , cutting  $AF$  in  $f$ ;  $f$  is the centre of the circle. From  $f$ , draw  $fg$  parallel to  $de$ , and with centre  $f$ , and radius  $fg$ , describe a circle. This circle will be tangential to  $AB, AC$ , and have a radius  $fg$ , equal to  $DE$ .

## PROBLEM 21.

*Upon a given straight line, to describe a triangle, the angles at the base being  $40^\circ$  and  $50^\circ$ .*

Fig. 37.



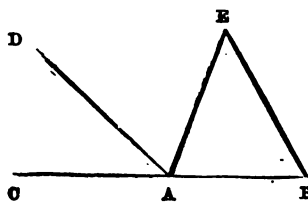
Let  $AB$  be the given line. At the point  $A$ , make the angle  $BAC$  equal  $50^\circ$  (Prob. 1). Again, at the point  $B$ , make the angle  $ABC$  equal  $40^\circ$ . The intersection  $C$ , of the lines thus drawn, will give the triangle  $ABC$  required.

Since the angles of a triangle are equal to two right angles or  $180^\circ$ , the triangle  $ABC$  may be constructed thus:—Make the angle  $BAC$  equal  $50^\circ$ , and from  $B$ , let fall  $BC$  perpendicular to  $AC$ ; the angle  $ABC$  will contain  $40^\circ$ . The same result will be obtained by making the angle  $ABC$  equal  $40^\circ$ , and drawing from  $A$ ,  $AC$  perpendicular to  $BC$ .

## PROBLEM 22.

*On a given straight line, to describe an isosceles triangle having a given vertical angle.*

Fig. 38.



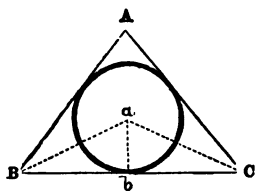
Let  $AB$  be the given line. Produce  $AB$  to  $C$ , and make the angle  $CAD$  equal to the given angle. Bisect the angle  $BAD$  by  $AE$  (Prob. 2). At  $B$ , make the angle  $ABE$  equal to the angle  $BAE$ ; and  $AEB$  will be a triangle whose vertical angle  $E$ , is equal to the given angle  $CAD$ .

PROBLEM 23.

*To inscribe a circle in a triangle.*

*Def.* A right-lined figure is *inscribed* in a circle, or the circle circumscribes the figure, when the angular points of the figure are in the circumference of the circle.

Fig. 39.

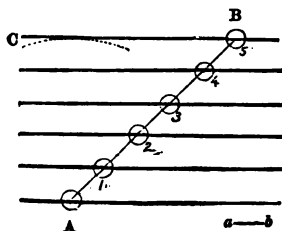


radius  $a b$ , describe a circle; this circle will be inscribed in the triangle.

PROBLEM 24.

*Through the points of division of a given divided line, to draw lines parallel to each other at a given distance apart.*

Fig. 40.



Let  $A B$  be the given line, divided into 5 equal parts in the points 1, 2, 3, etc., and  $a b$  the given distance at which the parallel lines are to be drawn.

With centre  $A$ , and radius equal 5 times  $a b$ , describe a circle. From  $B$ , draw  $B C$  a tangent to this circle (Prob.

10, Fig. 24). Through the points 4, 3, 2, etc., draw lines parallel to  $C B$ .

*Obs.* If it were required to divide a straight line 5 inches long into 8 equal parts, and, through the points of division,

to draw parallel lines  $\frac{1}{8}$ -inch apart, we should make  $A B$  equal 5 inches, and, having divided it into 8 equal parts, we should proceed as in the problem, *i.e.*, we should describe a circle with a radius of 8 times  $\frac{1}{8}$ -inch (4 inches), etc.

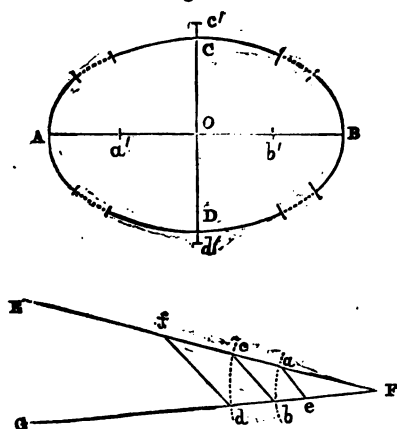
### PROBLEM 25.

*To describe an ellipse.*

An ellipse, like the circle, is a plane figure contained by one line called the circumference. It has two diameters, called axes, and distinguished as the *transverse* and *conjugate*. There are various ways of describing it, but we shall confine ourselves to two, and the student can use either at pleasure.

*First Method.* Let  $A B$  be the transverse, and  $C D$  the conjugate axis of

Fig. 41.



the required ellipse. Draw two lines  $F E$ ,  $F G$ , making any angle. With centre  $F$ , and radius  $O C$  or  $O D$ , the semi-conjugate axis, describe an arc  $a b$ ; and with centre  $F$ , and radius  $O A$  or  $O B$ , the semi-transverse axis, describe an arc  $c d$ . Join  $c b$ , and from  $a$ ,  $d$ ,

draw  $a e$ ,  $d f$ , parallel to it. From  $A$ ,  $B$ , set off  $A a'$ ,  $B b'$

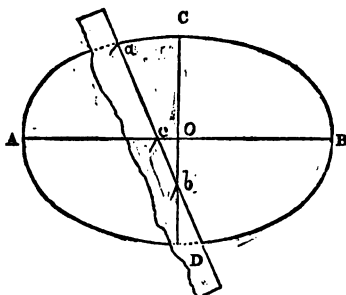
equal to  $F e$ ; and from  $c, D$ , set off  $c d', D d'$ , equal to  $F f$ . With centres  $a', b'$ , and radius  $a' A, b' B$ , we describe a portion of the ellipse; and with centres  $c', d'$ , and radius  $c' D, d' c$ , we describe another portion.

The parts of the circumference thus drawn, are shown by the continuous lines; the remaining parts can be readily sketched in, as shown by the dotted lines, or the ellipse may be completed by the construction given in second method.

The foregoing method will be found sufficiently accurate for all practical purposes, while it possesses the advantage of economizing time.

*Second Method.* Let  $A B$  be the transverse, and  $c D$  the

Fig. 42.



conjugate axis of the ellipse, as in the last case. Take a slip of paper, and, on its straight edge, set off  $a b$  equal  $o b$  half the transverse axis; and  $a c$  equal  $o c$  half the conjugate axis.

Apply the paper so that the point  $c$  fall on  $A B$ , the transverse axis,

and  $b$ , on  $c D$ , the conjugate axis. By keeping the points  $c, b$ , on  $A B, c D$ , and moving the paper round, we can mark at  $a$  any number of points through which the circumference of the ellipse must be described.

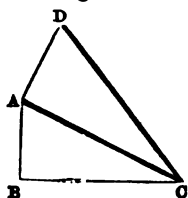
The principle involved in this method, possesses the recommendation of being that which is embodied in the construction of instruments used for describing ellipses. The student will find that the ellipse is a figure by no means easy to ink in neatly. Great assistance may be

derived, however, from French curves, which may be purchased at any Artist's Colourman's. The operation is greatly facilitated by employing the first method, as a considerable portion of the ellipse can be drawn by means of the bow-pen.

### PROBLEM 26.

*To determine by geometrical construction, lines which shall represent  $\sqrt{5}$ ,  $\sqrt{6}$ , or  $\sqrt{\frac{1}{2}}$ ,  $\sqrt{\frac{1}{3}}$ .*

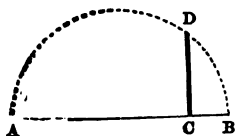
Fig. 43.



1°. In the right-angled triangle  $ABC$ , let  $AB$ ,  $BC$ , contain respectively, 1 and 2 units. Then by Eu. I., 47,  $-(AB)^2 + (BC)^2 = (AC)^2$ , or  $(AC)^2 = 1^2 + 2^2 = 5$ ; therefore,  $AC = \sqrt{5}$ .

From  $A$ , erect a perpendicular  $AD$  equal  $AB$ . Then  $(AD)^2 + (AC)^2 = (DC)^2$ , or  $(DC)^2 = 1 + 5 = 6$ ; therefore,  $DC = \sqrt{6}$ . By a similar construction, lines representing other quantities may be found.

Fig. 44.



2°. Make  $AB$  equal to  $\frac{1}{2}$  of a given unit, and upon it describe a semi-circle. From  $A$ , set off  $AC$  equal to the given unit, and from  $C$ , draw  $CD$  perpendicular to  $AB$ , cutting the semi-circle in  $D$ . Then, by Prob. 13,  $\sqrt{AC \times CB} = CD = \sqrt{1 \times \frac{1}{2}} = \sqrt{\frac{1}{2}}$ .

From the above, it will be readily seen how to determine a line which shall represent  $\sqrt{\frac{1}{3}}$ .

# TRANSFORMATION OR REDUCTION OF PLANE FIGURES.

## PROBLEM 27.

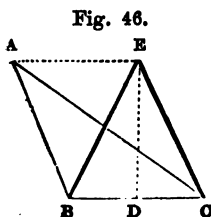
*To make a rectangle equal to a given triangle.*



Let  $\triangle ABC$  be the given triangle. Bisect  $BC$  in  $D$ , and through  $A$ , draw an indefinite straight line parallel to  $BC$ . From  $B, D$ , erect perpendiculars to  $BC$ , cutting the parallel, drawn from  $A$ , in  $E, F$ ;  $BDEF$  is the rectangle required. For the reason of the construction here employed, see Euc. I., 41.

## PROBLEM 28.

*To make an isosceles triangle equal to any other triangle.*



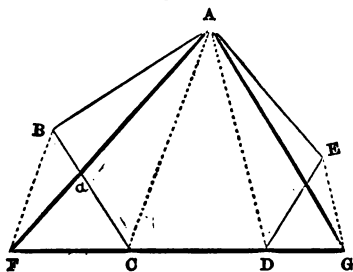
Let  $\triangle ABC$  be the given triangle. Draw an indefinite straight line  $AE$  parallel to  $BC$ . Bisect  $BC$  in  $D$ , and from  $D$  erect the perpendicular  $DE$ . Join  $BE, CE$ , and  $\triangle BCE$  will be the isosceles triangle required. (See Euc. I., 37.)

## PROBLEM 29.

*To make a triangle equal to a given rectilineal figure.*

Let  $ABCDE$ , the given figure, be an irregular pentagon.

Fig. 47.



Produce  $CD$  indefinitely both ways. Join  $AC$ , and draw  $BF$  parallel to it, meeting  $CD$  produced in  $F$ . Again, join  $AD$ , and draw  $EG$ , parallel to it, meeting  $CD$  produced, in  $G$ . Join  $AF$ ,  $AG$ . Then by Euc. I., 37, the triangle  $CBF$  is equal to the

triangle  $ABF$ . Take away  $ABF$  which is common to both triangles, and the triangle  $ABa$  will be equal to  $aFC$ . Substitute the triangle  $aFC$  for  $ABa$ ; then  $AFDE$  will be equal to  $ABCDE$ . The same reasoning applies to the triangles  $DEG$ ,  $AEg$ ; and, therefore, the triangle  $AFG$  is equal to  $ABCDE$ .

Since the area of a triangle is equal to the product of the base multiplied by half its perpendicular height (see Prob. 27 and Euc. I., 41), we can thus find the area of a rectilineal figure of any number of sides.

*Obs. 1.* This problem has a practical application, in fortification, in reducing the profile of a parapet to a triangle, by the area of which the dimensions of the ditch are regulated.

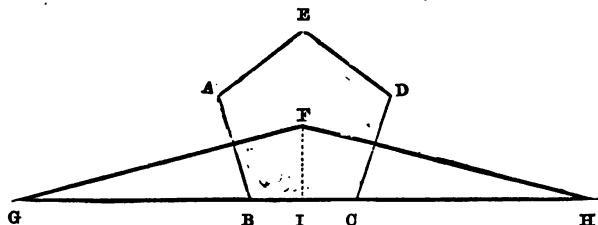
*Obs. 2.* The area of a profile or any given rectilineal figure, can be found by reducing it to triangles and trapeziums. See note on the "Calculation of the equality of Deblai and Remblai," page 215 of Captain Lendy's work on Fortification.

## PROBLEM 30.

*To make a triangle equal in area to a regular pentagon.*

Let  $ABCDE$  be the given pentagon. Produce  $BC$  both

Fig. 48.



ways, and make  $GH$  equal to the sum of the sides of the pentagon, *i.e.*, 5 times  $BC$ . Join  $G, H$ , to the centre  $F$ , and the triangle  $FGH$  will be equal to the given pentagon.

By drawing lines from the angles of the polygon to the centre, the figure will be divided into 5 identical triangles whose common altitude is  $FI$ . Therefore, since  $GH$  is equal to 5 times  $BC$ , and  $FI$  is the altitude of the required triangle, it is evident that the area of the triangle  $FGH$  is equal to the sum of the areas of the triangles of the figure.

## PROBLEM 31.

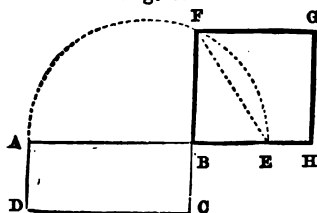
*To make a rectangle equal to a given regular pentagon.*

Make  $GH$  (see last figure) equal to half the sum of the sides of the pentagon, and upon it construct a rectangle having an altitude equal to  $FI$ . The area of this rectangle will be equal to the area of the given polygon.

## PROBLEM 32.

*To make a square equal to a given rectangle.*

Fig. 49.



Let  $ABCD$  be the given rectangle. Produce  $AB$  to  $E$ , making  $BE$  equal to  $BC$ . Find (Prob. 13)  $BF$ , the mean proportional between  $AB$ ,  $BE$ . The square  $BFGH$  described upon  $BF$ , is equal to the given rectangle.

## PROBLEM 33.

*To make a square equal to any part of a given square.*

Let  $AE$  (see last Fig.) be the side of the given square, and let the area of the required square be two-ninths of the square described upon  $AE$ . Upon  $AE$ , describe a semi-circle, and find, by means of the sector,  $BE$  equal two-ninths of  $AE$ . Draw  $BF$  perpendicular to  $AE$ , and join  $FE$ . The square described upon  $EF$  will be equal to two-ninths of the square described upon  $AE$ ; for  $BF$  being a mean proportional between  $AB$ ,  $BE$ , we have (assuming  $AE = 9$ )  $BF = \sqrt{2 \times 7} = \sqrt{14}$ , and  $(FE)^2 + (BE)^2 = 14 + 4 = 18 = (AE)^2$ . Now,  $18 = \frac{2}{9}$  of 81, the area of the square on  $AE$ .

## PROBLEM 34.

*To make a square equal to the sum of two or more given squares.*

1. Make  $AB$ ,  $BC$  (see Fig. 43), equal to the sides of the two given squares. Join  $AC$ ; and the square on  $AC$  will be equal to the squares on  $AB$ ,  $BC$ .

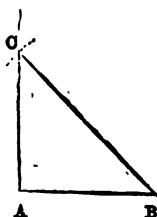
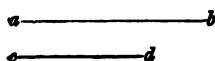
2. To make a square equal to the sum of three squares. Add two of them as before. We thus get  $\Delta c$ , the side of a square whose area is equal to the sum of two squares. Draw  $\Delta D$  perpendicular to  $\Delta c$ , and make it equal to the side of the third square. Join  $Dc$ ; then the square described on  $Dc$  will be equal to the sum of the three given squares.

*Obs.* By this problem a square may be made equal to the sum of any number of squares, or a square may be multiplied any number of times.

### PROBLEM 35.

*To make a square equal to the difference of two given squares.*

Fig. 50.



Let  $ab$ ,  $cd$ , be the sides of the given squares. Take  $\Delta B$  equal to  $cd$ . From  $\Delta$ , draw an indefinite straight line perpendicular to  $\Delta B$ , and with  $B$  as a centre, and  $ab$  as a radius, describe an arc cutting this line in  $C$ . Join  $Bc$ ; then  $(cb)^2 - (\Delta B)^2 = (\Delta c)^2$ , i.e., the square on  $\Delta c$  is equal to the difference of the squares on  $\Delta B$ ,  $Bc$ .

*Obs.* By this problem a triangle can be constructed, when one angle and two of its sides are given.

### PROBLEM 36.

*To draw a square equal to any given rectilineal figure.*

Reduce the given figure to a triangle (Prob. 29). Find a mean proportional between half the base and the perpendicular of this triangle. This mean proportional will be the side of the required square.

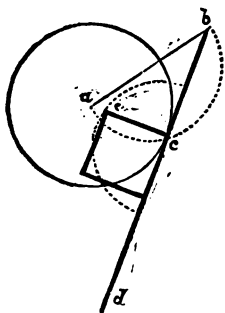
We shall now show how the foregoing problems are employed to solve more complicated ones.

### PROBLEM 37.

- (1) *Describe a circle with a radius of  $\cdot 4$  of an inch.*
- (2) *Draw a tangent to the circle from a point  $\cdot 3$  of an inch without it.*
- (3) *Join the given point and the point of contact of the tangent to the centre of the circle, and construct a square equal in area to the triangle thus formed.*

- (1) With centre  $a$  and  $\cdot 4$  inch as a radius, describe a circle. This answers the first condition of the problem.

Fig. 51.



- (2) Let  $b$  be the given point; then, by Prob. 10, draw  $db$  a tangent to the circle, touching it at  $c$ .

- (3) Join  $ab$ ,  $ac$ ; and find  $ce$  a mean proportional between  $ac$  and half the perpendicular  $cb$ , of the triangle  $abc$ . The square described upon  $ce$  is equal to the triangle  $abc$ .

*Obs.* 1°. The result will be the same in finding a mean proportional, whether we take half  $ac$  or  $cb$ . Thus, let  $a b = 5$ ,  $c b$ , 4, and  $a c$ , 3. Then, in the first case, the mean proportional between  $a c$  and half  $c b = \sqrt{3 \times \frac{1}{2}} = \sqrt{\frac{3}{2}}$ . Again, the mean proportional between  $c b$  and half  $a c = \sqrt{4 \times \frac{1}{2}} = \sqrt{2}$ .

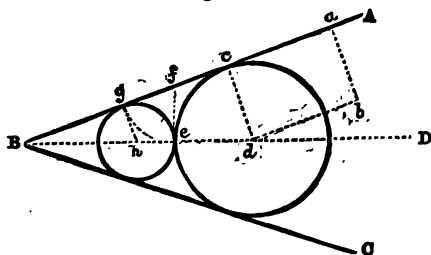
2°. If the triangle were not right-angled we should have to find a mean proportional between the base and half the perpendicular let fall upon the base.

## PROBLEM 38.

- (1) *Make an angle of  $40^\circ$  and bisect it.*
- (2) *Describe a circle, with  $\frac{1}{4}$  inch radius, tangent to both lines containing the angle.*
- (3) *Draw a second circle tangent to the first, and also to the lines containing the angle.*

Make the angle  $\angle B$   $40^\circ$  (Prob. 1) and bisect it by the

Fig. 52.



line  $BD$  (Prob. 2). Draw  $ab$  perpendicular to  $AB$ , and make it  $\frac{1}{4}$  inch long. For the remaining part of the construction, see Prob. 20.

To draw the second circle; from  $e$ , where the first circle cuts  $BD$ , draw  $ef$  perpendicular to  $BD$ . With centre  $f$  and radius  $fe$ , describe an arc, cutting  $AB$  in  $g$ ; and, from  $g$ , draw  $gh$  parallel to  $cd$  or  $ab$ , cutting  $BD$  in  $h$ . With centre  $h$  and radius  $hg$ , describe a circle. This circle will be tangent to  $AB$ ,  $BC$ , and touch the first circle at  $e$ .

## EXAMPLES.

*Note.* Those questions marked with an asterisk (\*) are taken from the Reports on the Military Examinations.

1. (a) Make an angle of  $70^\circ$  and bisect it. (b) Describe a circle, with 1 inch radius, tangent to both lines containing the angle. (c) Draw two other circles tangents to the first and also to the lines containing the angle. (Prob. 38.)

2.\* Draw a line 2 inches long perpendicular to a given line 3 inches long, from a point  $\frac{2}{3}$  of an inch from one end, by geometrical construction. (Prob. 4, Case 2.)

3. (a) Upon a line  $2\frac{1}{2}$  inches long, construct a rectilineal figure of seven sides. (b) Copy the foregoing on a scale of  $\frac{2}{3}$  the original, see Prob. 6, Fig. 14. Make  $a b$   $\frac{2}{3}$  of  $A B$ , and proceed as explained in the figure.

4. Reduce (a) last example, to a triangle of equal area. (Prob. 29.)

5.\* Construct an isosceles triangle equal to the sum of four squares of which the sides are  $\frac{1}{2}$ ,  $\frac{2}{3}$ ,  $\frac{1}{4}$ ,  $1\frac{1}{2}$  inches respectively (Probs. 34, 27). Having found a square equal to the sum of the four squares by Prob. 34, the isosceles triangle will have a base twice the side of this square, while its altitude will be equal to the side of the square. (See Euc. I., 41.)

6. (a) Divide a line 4 inches long into 7 equal parts. (b) Considering the divided line as a scale of equal parts, construct a triangle of which the sides are 5, 4, and 3 parts respectively (see Prob. 35, Fig. 50). Make  $A B$  equal 3 parts, and with  $A$ ,  $B$ , as centres and with radii equal to 4, 5 parts respectively, describe arcs intersecting in  $C$ . Join  $C A$ ,  $C B$ , and  $A B C$  will be the triangle required.

7. Given an arc of a circle, to find the centre of the circle. (Prob. 8.)

8. (a) Describe a circle with a radius of  $1\frac{1}{2}$  inches, and in it draw a regular heptagon. (b) Make a rectangle equal in area to the heptagon. (Probs. 8, 9, and 31.)

9. From a circle of 2 inches radius, cut off two segments containing angles of  $40^\circ$  and  $100^\circ$  respectively. (Prob. 15, Obs.)

10. (a) Construct a triangle of which two of its sides are

3 and 2·2 inches respectively, and one angle  $50^\circ$ . (*b*) Inscribe and circumscribe the triangle by a circle. (See Obs., Prob. 35, and Probs. 23 and 8.)

11. (*a*) Construct a regular octagon with a side of 2·2 inches. (*b*) Reduce the figure to a triangle. (Probs. 9, 30.)

12.\* Draw two circles with radii of ·92 and ·64 inch touching each other externally, and about them circumscribe a triangle whose angles shall be  $46^\circ$ ,  $62^\circ$ , and  $72^\circ$ .

13. (*a*) Construct an isosceles triangle upon a base of 2 inches, each of the angles at the base being  $70^\circ$ . (*b*) Make a rectangle equal in area to the triangle. (*c*) Make a square equal to half the rectangle. (Probs. 21, 27, and 32.)

14. (*a*) Make an isosceles triangle having a vertical angle of  $70^\circ$ . (*b*) Make a triangle equal to the above having an angle equal to a given angle.

(*a*) See Prob. 22. (*b*) Make the angle  $\angle CBA$  (see Prob. 28, Fig. 46) equal to the given angle, and join  $CA$ .

15.\* Describe two circles with radii of 2 inches and 1 inch respectively, tangent to one another, and inscribe a nonagon in the first. (Probs. 18 and 9.)

16. Explain the use of the *line of lines* on the Sector, and from a line 5·3 inches long cut off  $\frac{1}{4}$  of the length. (See Explanation of Sector.)

17.\* (*a*) Construct a triangle of which two sides are  $1\frac{1}{2}$  and  $2\frac{1}{2}$  inches long, and the included angle  $58^\circ$ ; circumscribe the triangle by a circle. (*b*) Join the centre of the circle and the given angle of the triangle, and upon this line as a base, construct a regular octagon. (*c*) Reduce the part of the octagon that lies outside the original triangle, to a triangle of equal area. (Probs. 35, 9, and 29.)

18. Construct a scale of chords and make an angle of  $70^\circ$ . (Prob. 1.)

19. (a) Divide a line 10 inches long in the proportion of the numbers 2, 2.5, 1.2, 2, 2.3. (b) Divide a line 4 inches long proportionally to the above. (Prob. 17.)

20. Find a fourth proportional to 2, 2.5, 3.3 inches.

21.\* Divide a straight line 5.3 inches long into 8 equal parts, and through the points of division draw parallel straight lines  $\frac{1}{2}$ -inch apart, making them alternately dotted and continuous. (Prob. 24.)

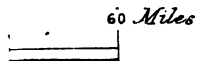
22.\* (a) Construct three squares, the areas of which are .81, 1.96, and 2.56 square inches. (b) Construct, geometrically, a fourth square, the area of which shall be equal to the sum of the areas of the other three.

*Note*—For additional problems on Practical Geometry, see Appendix at the end of the present volume.

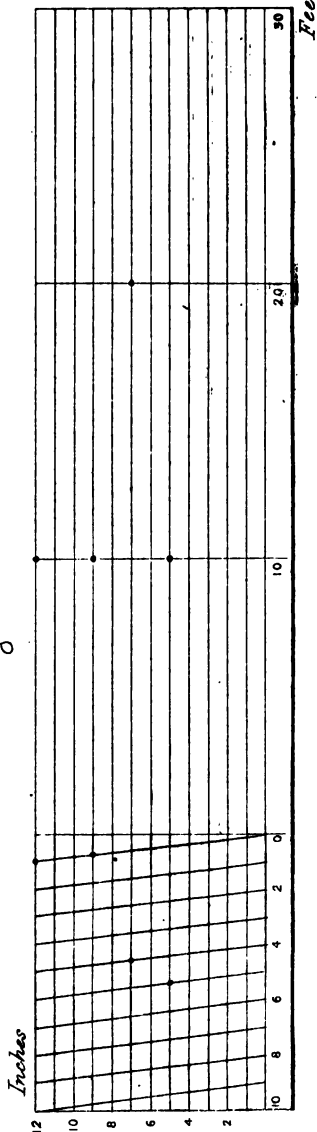
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## ON SCALES.

In the study of Engineering, Architecture, etc., it is necessary that the student should understand the construction and use of Scales. A drawing is said to be made to scale when its various parts bear a certain proportion to the parts of the object of which it is a representation. Let  $A B C D E$  (Fig. 14, Prac. Geom.) be a drawing of an object of which  $A B$  measures *actually* 120 yards. Now, by applying  $A B$  to a scale of equal parts, it will be found to measure  $1\frac{1}{2}$  or 1.2 inches, *i.e.*, a length of 120 yards is represented by a line 1.2 inches long; or, what is the same thing, a length of 100 yards is represented by a line 1 inch long; and the drawing is said to be made to a scale of 100 yards to 1 inch. To find the proportion of the drawing to the original object, bring 100 yards to inches. Number



No 8. Diagonal Scale of  $\frac{1}{4}$





of inches in 100 yards =  $100 \times 36 = 3600$  inches. Therefore, every line, in the original object, is 3600 times the line which represents it in the drawing; or, every line in the drawing is  $\frac{1}{3600}$  part of the corresponding line of the object.

Again, if a length of 60 feet be represented by 3 inches, 1 inch will represent 20 feet, and every line in the drawing will be  $\frac{1}{20}$ th of the corresponding line of the object. The fractions  $\frac{1}{3600}$ ,  $\frac{1}{20}$ , are called the *representative fractions* of the scales. *The representative fraction, then, shows the ratio of 1 inch to the number of units represented by 1 inch, whether of feet, yards, miles, etc.; or in other words, it expresses the relation between the drawing and the original object.* When the scale is constructed, the fraction should be written above it, or placed on the drawing when there is no scale.

From what has been said, it is hoped that the student will clearly understand what is meant by the *representative fraction*, to find which, he has only to reduce the number of units represented by 1 inch to inches. He is recommended to make an attempt to answer the following questions, without referring to the book.

(1) A drawing is made to the scale of 50 yards to 1 inch; required the representative fraction, or the relation between the drawing and the original object.

Here, the number of inches in 50 yards =  $50 \times 36 = 1800$ ; therefore, the representative fraction is  $\frac{1}{1800}$ , i.e., every line in the drawing is  $\frac{1}{1800}$ th part of the corresponding line of the object.

(2) A drawing is made to the scale of 1 mile to 1 inch required the representative fraction.

Here, the number of inches in 1 mile =  $1 \times 1760 \times 36 = 63360$ ; therefore, the representative fraction is  $\frac{1}{63360}$ .

(3) A drawing is made to the scale of 3 leagues to 1 inch; required the representative fraction.

Here, the number of inches in 3 leagues =  $3 \times 3 \times 1760 \times 36 = 570,240$ ; therefore, the representative fraction is  $\frac{1}{570240}$ .

### EXAMPLES.

1. Construct a scale to show 20 feet, when 1 inch represents 4 feet. Give the representative fraction.

Here, 1 inch represents 4 feet; therefore, to represent 20 feet it will take 5 times 1 inch = 5 inches; or, by proportion, thus:—

4ft. : 1in. :: 20ft. :  $x$  (the length required) = 5 inches.

Draw three parallel lines 5 inches long at equal distances apart (about the same as in the drawing). Divide this length (5 inches) into 2 equal parts (Prob. 6, Prac. Geom.) and the left hand part or primary division into 10 equal subdivisions. From the primary divisions on the lower line, draw perpendiculars to the top line; and from each subdivision, draw perpendiculars to the second line, except the centre one, which should be drawn half-way between the second line and the top line. (See Scale 1). Write 0 (zero) at the first primary division, 10 at the end of the subdivisions (shown at the left of the scale), and 10 at the primary division (shown at the right of the scale; 5 may also be written at the centre of the sub-divisions. The representative fraction  $\frac{1}{48}$  should be placed above the centre of the scale, and the units of measure on the right.

In inking in the scale, it will be observed that the top line is omitted, while the bottom line is drawn *thick*. Since No. 1 is in every respect, a model of the scales which

follow, the student will see the importance of thoroughly understanding it. It may be well to point out here, that in making a drawing to scale, the length of the scale should be such as to enable us to set off considerable distances. The number assumed (as 20 in the present scale) should be a number divisible by 10; and the quotient, resulting from the division of this number by 10, is the number of primary divisions into which the length of the scale should be divided (as 2 in the present scale).

The manner of using the scale is as follows:—Suppose it were required to take off 15 feet. Place one leg of the compasses at 10 on the right of the scale, and the other leg at 5, one of the sub-divisions, and the space included between these two points will contain 15 feet. In the same manner any number of feet from 1 to 20 may be taken off.

2. Construct a scale to show 70 yards, when 1 inch represents 9 yards. (No. 2.)

$$9\text{yds.} : 1\text{in.} :: 70\text{yds.} : x = 7\cdot77 \text{ inches.}$$

That is, if it takes 1 inch to represent 9 yards, it will take 7·77 inches to represent 70 yards. Therefore, make the scale 7·77 inches long, and divide it into 7 equal parts (the quotient of  $70 \div 10$ ). Divide the first primary division into 10 sub-divisions, and complete the scale as before, *i.e.*, by writing 10, 20, 30, 40, 50, 60, to the right of the scale, etc. The representative fraction is  $\frac{1}{3\frac{1}{4}}$ .

3. The distance between two places is 4 miles, and is represented on a plan by 1·5 inches; construct the scale to show furlongs, when 20 miles is the number assumed. (No. 3.)

Here, we have—

$$4 \text{ miles} : 1\cdot5\text{in.} :: 20 \text{ miles} : x = 7\cdot5 \text{ inches.}$$

That is to say, if it takes 1·5 inches to represent 4 miles, it will take 7·5 inches to represent 20 miles. Make the scale 7·5 inches long, divide it into 2 equal parts, and complete it as before. Divide the tenth sub-division (on the left hand) into 8 equal parts for furlongs; for since each sub-division represents 1 mile, the eighth part of one of these will represent 1 furlong.

The representative fraction of the scale is—

$$\frac{7.5}{20 \times 1760 \times 36} = \frac{1}{168960}$$

4. Construct a scale of 10 miles to 1 inch, to measure spaces of 1000 yards. Assume 70 miles. (No. 4.)

10 miles : 1 in. :: 70 miles :  $x$  = 7 inches.

Make the scale 7 inches long, and divide into 7 equal parts. Each primary division represents 10 miles, which, divided by 1000, will give  $17\frac{1}{2}$  parts. Therefore, divide the first primary division into this number of parts, and complete the scale as shown in the figure. All distances less than 1000 yards must be taken by the judgment only.

The distance between the dots shows 10 miles 1500 yards.

5. Draw a scale of 1 mile to 1 inch, to show furlongs. (No. 5.)

Make the scale the required length, suppose 5 inches. Divide it into 5 equal parts, and the first primary division into 8 sub-divisions for furlongs. Write *miles* on the right, and *furlongs* on the left, as shown on the scale. (No. 5.)

It will be observed that the first primary division in Scales 4 and 5 is not divided into 10 equal parts as in the preceding scales. Though, as a rule, the decimal notation should be adopted, there are special cases where an exception must be made, as in the examples referred to. In fact,

the construction of scales is a subject which demands both judgment and ingenuity. Suppose it were required to construct a scale of 40 feet to 1 inch to measure single feet, and that 240 feet is to be the number shown. We should make the scale 6 inches long; divide it into 6 equal parts, and the first primary division into 4 sub-divisions, to measure spaces of 10 feet. We should then sub-divide the the first sub-division into 10 equal parts. In figuring the scale, we should write 0 (zero) on the extreme left, and then the numbers 10, 20, 30, 40, which will include one primary division. From this point, the numbers will be 80, 120, 160, 200, and 240.

### EXAMPLES.

(a) Construct a scale of 1 foot to 1 inch to measure inches. Show 9 feet.

(b) Draw a scale of 1 league to 1 inch to measure miles. Show 8 leagues.

Since in the representative fraction is expressed the number of inches represented by 1 inch, the scale may be constructed when this fraction is given. Suppose a scale of feet is required, the representative fraction being  $\frac{1}{48}$ , as in No. 1. By proportion we have—

$$48 : 1 :: 20 \times 12 : x = 5 \text{ inches,}$$

the length of the scale. That is to say, as 48 inches (the denominator of the fraction) is to 1 inch (the numerator), so is the number of inches in the number of units assumed to  $x$ , the required length of the scale.

Again, in No. 2 we have—

$$324 : 1 :: 70 \times 36 : x = 7.77 \text{ inches,}$$

the length of the scale.

## COMPARATIVE SCALES.

One scale is said to be *comparative* to another, when the distances measured by the one can be measured by the other. For example, if we have a scale of miles to which a drawing is made, we can construct another scale by which the distances from place to place can be measured in furlongs. The scale of furlongs would then be comparative to the scale of miles. In making one scale comparative to another, therefore, we must of necessity have two units of measure, as an inch and a foot, a foot and a yard, a mile and a chain, etc. Suppose it is required to make a scale of yards comparative to No. 1. By proportion we have—

$$4\text{ft.} : 1\text{in.} :: 3\text{ft.} : x = \frac{3}{4}\text{-inch.}$$

That is to say, if 1 inch represents 4 feet, it will take three-fourths of an inch to represent 1 yard or 3 feet; and, therefore, the number of inches to represent 60 feet (20 yards) will be  $\frac{3 \times 20}{4} = 15$  inches; because it will take 20 times the length of scale to represent 60 feet (20 yards) that it takes to represent 3 feet; but it takes  $\frac{3}{4}$ -inch to represent 3 feet, therefore it will take 20 times  $\frac{3}{4}$ -inch = 15in., to represent 60 feet.

Make the scale 15 inches long, divide it into 2 primary divisions, and the first primary division into 10 subdivisions to measure single yards. The scale is completed in every respect as before.

The proportion may also be stated thus :—

$$12 : 5 :: 36 : x = 15 \text{ inches.}$$

That is to say, as 12, the number of inches in 1 foot, is to 5 inches, the length of the scale, so is 36, the number of inches in 1 yard, to  $x$ , the required length of the scale.

Again, thus :—

$$20 : 5 :: 20 \times 3 : x = 15 \text{ inches.}$$

That is to say, as 20 feet is to 5 inches, the length of the scale, so is 20 yards brought to feet, to  $x$ , the required length of the scale.

6. Draw a scale of kilomètres comparative to No. 3. The kilomètre = 1093·63 English yards. (Scale No. 6.) The number of miles represented in Scale No. 3 is 20 miles. By proportion we have—

$$20 \times 1760 : 7\cdot5 :: 20 \times 1093\cdot63 : x = 4\cdot66 \text{ inches.}$$

That is to say, as the number of yards in 20 miles is to 7·5 inches, the length of the scale, so is the number of yards in 20 kilomètres to  $x$ , the required length of the scale.

Make the scale 4·66 inches long; divide it into two equal parts, and the first primary division into 10 equal parts. Complete the scale as before, and write *kilomètres* on the right. (See No. 6.) The representative fraction is—

$$\frac{4\cdot66}{20 \times 1093\cdot63 \times 36} = \frac{1}{168972}$$

7. In examining a French plan, I find only a scale of decimètres, 10 to 1 inch; supply a comparative scale of English feet, the decimètre being equal to ·327 English foot. Show 20.

Number of English feet in 10 decimètres =  $\cdot327 \times 10$  = 3·27. Then, by proportion, we have—

$$3\cdot27\text{ft.} : 1\text{in.} :: 20\text{ft.} : x = 6\cdot11 \text{ inches.}$$

That is to say, as the number of feet in 10 decimètres is to 1 inch, the length which represents it, so is 20 feet to the required length of the scale. Therefore, make the scale 6·11 inches long, divide it into 2 equal parts, and the first primary division into 10 equal parts. Complete the scale as before, and write *feet* on the right. (See No. 7.)

*Obs.* It is not necessary that the comparative scale should represent the same number of units as the original scale. If this were the case, the scale would frequently be inconveniently long. For example, if the number assumed in the last question had been 80 instead of 20, the scale would have been more than 24 inches long. Perhaps we shall be more clearly understood by giving the question (No. 7) in another form. Thus:—In examining a French plan, I find that 100 décimètres are represented on a scale 10 inches long; supply a comparative scale of English feet, the décimètre being equal to  $\cdot 327$  English foot.

Number of feet in 10 décimètres =  $\cdot 327 \times 10 = 3\cdot 27$  feet. Therefore, as—

$3\cdot 27\text{ft.} : 1\text{in.} :: 100\text{ft.} : x = 30\cdot 58 \text{ inches;}$   
or thus:—The number of feet in 100 décimètres =  $\cdot 327 \times 100 = 32\cdot 7$  feet. Therefore, as—

$32\cdot 7\text{ft.} : 10\text{in.} :: 100\text{ft.} : x = 30\cdot 58 \text{ inches.}$

Now, 30·58 inches, the length of the scale, bears the same proportion to 100 feet, the number of units represented, as 6·11 inches, the length of the scale (see No. 7), bears to 20 feet, the number of units represented. The student, therefore, has only to bear in mind, that the number of units represented on the original scale need not necessarily be represented on the new scale.

### DIAGONAL SCALES.

Diagonal Scales are used for taking off more minute distances than can be done by an ordinary scale.

8. Draw a Diagonal scale of 7 feet to 1 inch to show inches. Assume 40. (No. 8.)

$7\text{ft.} : 1\text{in.} :: 40\text{ft.} : x = 5\cdot 71 \text{ inches.}$

Draw a line 5·71 inches long; divide it into 4 equal parts, and the first primary division into 10 equal parts. Each of these sub-divisions will represent 1 foot, as in the preceding scales. Parallel to the first line, draw 12 lines about the same distance apart as shown in the scale. Through the points of divisions 10, 0, 10, 20, 30, draw perpendiculars. (See Scale No. 8.) Number the points on the first primary division 2, 4, 6, 8, 10, and those on the perpendicular drawn through 10, on the extreme left of the scale, 2, 4, 6, 8, 10, 12. Join 12 to the ninth sub-division, and through the points 8, 7, 6, 5, 4, 3, 2, 1, 0, draw parallels (diagonals).

Below the line first taken, draw another line at a greater distance apart than the others; make it a thick line and write *feet* and *inches* on the extreme right and left of the scale. We have thus constructed a diagonal scale by which we can measure feet and inches. The distance between the dots on the upper line shows 11 feet, while the distance between the second dots (on the parallel drawn through 9) shows 10 feet 9 inches. Suppose it is required to take off 15 feet 5 inches. Place one leg of the dividers at the point where the diagonal drawn through 5 intersects the parallel drawn through 5; and the other leg at the point where the same parallel intersects the perpendicular drawn through 10. The distance is shown by the dots on the parallel drawn through 5. The distance between the dots on the parallel drawn through 7 shows 24 feet 7 inches.

Again, let it be required to construct a scale of 10 miles to 1 inch, showing furlongs diagonally.

Take a line any convenient length, say 6 inches; divide into 6 equal parts, and the first inch to the *left* into 10 equal parts. Since the inch represents 10 miles each of the sub-divisions will represent 1 mile. Now, to show furlongs, we

want the one-eighth part of one of these sub-divisions. Therefore, draw 8 lines parallel to the line first drawn, and about the same distance apart as shown in Scale No. 8. Complete the scale as shown at No. 8.

Let it be required to construct a scale of 1 furlong to  $1\frac{1}{2}$  inches, showing yards diagonally.

Take a line any convenient length, say 6 inches long. Divide this line into 4 equal parts, each part of which will contain  $1\frac{1}{2}$  inches and represent 1 furlong. Now, the number of yards in 1 furlong = 220, which resolved into factors, =  $11 \times 20$ . Therefore, divide the first division into 20 sub-divisions, each of which will represent 11 yards ( $\frac{11}{20}$ "); and to measure single yards, we have merely to draw 11 lines parallel to the first line—completing the scale as before.

It is hoped that the foregoing examples will render the principles, involved in the construction of a diagonal scale, sufficiently obvious without further explanation.

### EXAMPLES.

*Note.* Those questions marked with an asterisk (\*) are taken from the Reports on the Military Examinations.

1. Construct a scale of 30 feet to 1 inch, to show single feet.
2. Construct a scale of yards of  $1\frac{1}{8}$ . Show 50 yards.
3. The distance between two places is 10.6 miles and measures on the scale 2 inches; draw the scale. Show 30 miles.
- 4.\* An Englishman wishing to examine a Spanish plan, finds only a scale of Spanish palms, 20 to 1 inch; supply him with a corresponding scale of English feet, taking the palm as .684 English foot. Show 50 feet.
5. Make a diagonal scale of 10 feet to  $1\frac{1}{2}$  inches, show-

ing inches diagonally, and explain the principles of construction.

6. Draw a diagonal scale of 10 miles to 1 inch, to show furlongs. Assume 70 miles.

7. Draw a diagonal scale to show 1000ths of 12 inches. Take a line 12 inches long, divide it into 10 equal parts, and sub-divide the first primary division into 10 equal parts. Then will each primary division be  $\frac{1}{10}$  of 12 inches, and each sub-division  $\frac{1}{10}$  of  $\frac{1}{10}$  =  $\frac{1}{100}$  of 12 inches. Draw 10 lines parallel to the line first drawn, and complete the scale as shown at No. 8. The diagonals will show  $\frac{1}{10}$  of  $\frac{1}{10}$  =  $\frac{1}{100}$ .

8. On a scale, 60 Russian versts measure 7.5 inches; supply a comparative scale of English miles, taking the verst as 1167 yards. Show 40 miles.

9.\* Draw scales of  $\frac{1}{10}$  to represent English feet, metres and Greek cubits, 1 metre being = 3.27 feet, and 1 Greek cubit = .45 metre.

10.\* Draw scales of  $\frac{1}{1000}$  to show English miles and Russian versts. Verst = 1166.68 yards.

11.\* The distance between London and Chatham is 30 miles, and measures on a map 18.3 inches; draw the scale of the map divided into miles and furlongs, and mark the representative fraction.

12.\* Draw scales of  $\frac{1}{10}$  to measure Belgian and Chinese feet. 1 Belgian foot = .90466 English foot = .8616 Chinese foot.

13. Draw a diagonal scale of 1 foot to 1 inch, to show eighths of an inch.



*Table showing some of the principal Units of Linear Measure in terms of English Measure.*

PLACE.	UNITS.	FEET.	YARDS.	MILES.
Austria	Zoll (12 Linien) ... ..	·0864	·0288	...
"	Fuss or Schuh ... ..	1·03704	·34568	...
"	Elle ... ..	...	·85289	...
"	Klafter (6 Fuss) ... ..	...	2·0741	...
"	Ruthe (10 Fuss) ... ..	...	3·4568	...
"	Meile ... ..	...	5863·3	3·3312
"	Meile (Geographische) ... ..	...	8100·8	4·6026
Belgium	Fuss (11 Zolle) ... ..	·90466	·30155	...
"	Elle ... ..	...	·74845	...
"	Verge ... ..	...	4·9255	...
"	Meile ... ..	...	4860·833	2·7641
France	New Measures	Milimètre ... ..	·0033	·0011
"		Centimètre ... ..	·0327	·0109
"		Decimètre ... ..	·3279	·1093
"		Mètre ... ..	...	1·0936
"		Kilomètre ... ..	...	1093·63
"		Myriamètre ... ..	...	10936·33
"		Pouce (12 Lignes) ... ..	·08864	·02955
"		Pied ... ..	1·06571	·35523
"		Toise (6 Pieds) ... ..	...	2·13142
"		Brasse Marine (5 Pieds) ... ..	...	1·77618
"	Old Measures	Lieue de Poste (2,000 Toises) ... ..	...	4262·84
"		Lieue Marine (·05 Degre) ... ..	...	6275·6
"		Mille Marine (·333 lieue=1 min.) ... ..	...	2025·2
"		...	...	1·1506
Hanover	Fuss ... ..	·9579	·3193	...
Holland	Fuss ... ..	·9288	·3096	...
"	Meile ... ..	...	6404·12	3·6387
Naples	Palma ... ..	·8628	·2876	...
"	Canna ... ..	...	2·3008	...
Portugal	Pié ... ..	1·08266	·3608	...
"	Palmo ... ..	·7171	·239	...
Prussia	Fuss (Rhenish Foot) ... ..	1·0297	·3432	...
"	Elle ... ..	...	·7293	...
"	Schritt ... ..	...	·8234	...
"	Klafter or Faden (6 Fuss) ... ..	...	2·0594	...
"	Ruthe (12 Fuss) ... ..	...	4·1188	...
"	Meile ... ..	...	8272	4·7
Rome	Piède ... ..	·9665	·3222	...
Russia	Archine ... ..	...	·7777	...
"	Sachine ... ..	...	2·3332	...
"	Verst ... ..	...	1166·6	...
Saxony	Fuss ... ..	·9294	·3098	...
"	Meile ... ..	...	9923·326	5·6332
Spain	Pulgado ... ..	·0761	·0254	...
"	Palmo ... ..	·6849	·2283	...
"	Pié (Castilian) (12 Pul.) ... ..	·9132	·3044	...
"	Vara (3 Pies) ... ..	...	·9132	...
Sweden	Foot (10 inches) ... ..	·974	·3246	...
"	Alner ... ..	1·948	·6493	...
"	Mile ... ..	...	11688	6·6412

## ORTHOGRAPHIC PROJECTION.

1. Orthographic Projection is that branch of geometrical drawing, which enables us to represent objects on paper in such a manner that, from a drawing, the objects themselves may be constructed.

These representations are made upon planes usually conceived to be at right angles to each other. It is indispensably necessary, in the first place, that the student should have a clear conception of these planes, which we shall proceed at once to explain and illustrate.

2. The planes of projection are distinguished as the *horizontal* plane, and the *vertical* plane, and may be familiarly illustrated by the floor and walls of a room—the floor representing a horizontal plane, and the walls so many vertical planes.

3. The planes intersect each other in a line variously termed the *intersecting line*, the *ground line*, the *base line*, and the *line of level*. As referred to the room, this line is that in which the floor and walls meet each other.

4. If the reader will hold his box of instruments in a horizontal position, so as to view, first, its length, and then its breadth, the views will be termed respectively, a *side view*, and an *end view*. Again, if we look at a building, so as to see its front, we have a *side* or *front view*; and if we view it, so as to see the distance it projects from rear to front, we have an *end view*.

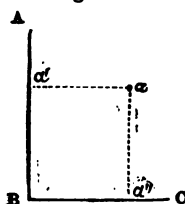
To the foregoing representations, we apply the term *elevation*, and express which elevation by the terms, *front*, *side*, and *end*.

5. Now, let the reader place the box on the table, and look *down* at it. This view is called a *plan*. An *elevation*, then, may be defined to be that representation of an object, which shows its altitude from a horizontal plane; and a *plan* that representation of an object as it is seen on a horizontal plane when viewed from *above*; or, as a familiar illustration, we may say that a map of a country or town is a *plan*, while the mountains or buildings it contains, when seen in relief, are termed *elevations*.

6. These terms, plan and elevation, derive their names from the planes of projection upon which the object is represented.

The planes of projection, upon which objects are represented, are, as has been explained, the horizontal plane, and the vertical plane—*plan* being referable to the horizontal, and *elevation*, to the vertical plane.

7. Let  $\triangle ABC$  be an end view of a sheet of paper folded as to form a right angle at  $B$ . Now,



$\triangle ABC$  may be considered as an end view of the planes of projection,  $AB$  being the vertical, and  $BC$  the horizontal plane. Let  $a$  be a point in space whose projections upon  $AB$ ,  $BC$  are required.

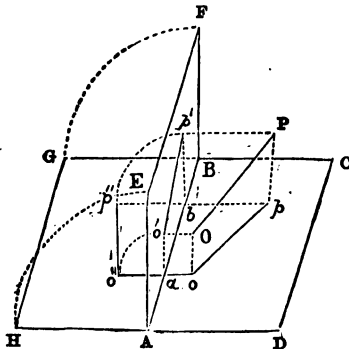
8. The projection of a point upon a plane is the foot of a perpendicular let fall from the point upon the given plane.

From  $a$ , therefore, draw  $aa'$ ,  $aa''$ , respectively perpendicular to  $AB$ ,  $BC$ ; the points  $a'$ ,  $a''$ , where these perpendiculars meet the given planes, are the projections of  $a$  in space.

9. The line which projects a point upon a plane, is called the *projector* of that point;  $aa'$ ,  $aa''$  are the projectors of the point  $a$ .

10. When the projections of a point are given, the point itself may be determined, for it is the point of intersection of the projectors of the point.

Fig. 2.



11. We shall now show how the projections of a line upon the two planes of projection are obtained.

Let  $ABCD$  be the horizontal, and  $AEFB$  the vertical plane of projection; also let  $oP$  be the position of a line in space.

The plan of  $P$ , or its projection upon the plane  $ABCD$ , is the foot of the perpendicular let fall from  $P$  upon the given plane (8). Let  $p$  be its projection. The plan of  $o$  is the foot of the perpendicular let fall from  $o$ . Let  $o$  be its plan. Join  $op$ ;  $op$  is the plan, or projection of  $oP$  upon the horizontal plane.

Again, the projection of  $P$  upon the vertical plane, is the foot of the perpendicular drawn from the point to that plane.

Let  $p'$  be its projection. In the same manner we obtain  $o'$ , the projection of  $o$ . Join  $o'p'$ ;  $o'p'$  is the elevation, or projection of the line  $oP$  upon the vertical plane.

*Obs. 1°.* Having determined  $o$ ,  $p$ , the projections of  $o$ ,  $P$  upon the horizontal plane, the elevation  $o'p'$  is found thus:—Draw  $oa$ ,  $pb$  at right angles to the plane  $AEFB$ , meeting the intersecting line,  $AB$ , of the two planes, in  $a$  and  $b$ . From  $a$ ,  $b$ , draw lines parallel to  $oo$ ,  $Pp$ ; then, the intersections  $o'$ ,  $p'$ , of these lines with the perpendiculars let fall from  $o$ ,  $P$ , will be the projections sought.

*Obs. 2°.* It will be readily seen how, from the elevation  $o'p'$ , we obtain the plan  $op$ —the operation being the converse of that shown above.

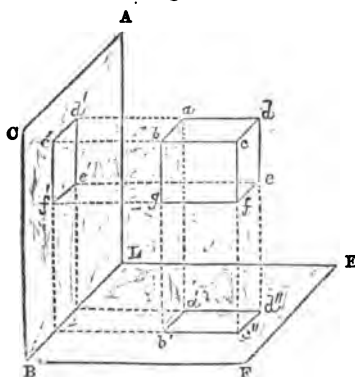
12. The plane which projects a line upon either plane of projection, is called the projecting plane of that line.

The plane  $oppo$ , passing through the line  $op$ , projects that line upon the plane  $ABCD$ , while the plane  $opp'o$  projects it upon the plane  $AEFB$ .

It will be observed that, in each case, the projecting plane is perpendicular to the planes of projection.

13. Since every solid is bounded by planes or surfaces, and surfaces by lines, and lines by points, if the projection of a point and a line, upon the two planes of projection be thoroughly understood, the projection of a solid will be readily comprehended.

Fig. 3.



14. Let  $abcd$ , etc., be the position, in space of a regular solid; and let  $ABCD$ ,  $AEFB$  be the planes of projection.

The projection of the solid upon the two planes, is simply a repetition of the preceding problems. The plan of  $c$  will be the foot of a perpendicular let fall from  $c$  upon the

plane  $AEFB$  (8). We thus obtain  $c'$ . In the same manner, we obtain  $b'$ . Join  $b'c'$ ;  $b'c'$  is the plan of the line  $bc$ . In the same way, we find  $a'd'$ , the plan of  $ad$ . Join  $a'b'$  and  $c'd'$ ; and  $a'b'c'd'$  is the plan, or projection of the given solid upon the horizontal plane of projection.

The intersections of the perpendiculars from the points  $c$ ,  $d$ ,  $e$ ,  $f$ , with the plane  $A C B L$ , will give the elevation, or projection of the solid upon the vertical plane.

The plane  $b b' c' c$ , passing through the line  $b c$ , projects that line upon  $B L E F$ . Again, the plane  $c c' d' d$ , passing through the line  $c d$ , projects that line upon  $A C B L$ .

15. The line  $b c$ , and all lines parallel to it, are parallel to the horizontal plane of projection; and the line  $c f$ , and all lines parallel to it, are parallel to the vertical plane of projection. Again, the line  $b c$ , and all lines parallel to it, are projected upon the horizontal plane of projection in lines equal and parallel to themselves; and the same remark applies to the projections of the line  $c f$ , and to all lines parallel to it, upon the vertical plane of projection.

16. The line  $c f$ , and all lines parallel to it, are perpendicular to the horizontal plane of projection, and are projected on that plane in points. Again, the line  $c b$ , and all lines parallel to it, are perpendicular to the vertical plane of projection, and are projected on that plane in points.

17. From the foregoing we establish:—

(a) That when a line is parallel to the horizontal and vertical plane, its projections are lines parallel to the base line  $B L$ , and equal in length to the original line. The projections  $c' d'$ ,  $c'' d''$ , of the line  $c d$ , are parallel to  $B L$ , and equal in length to  $c d$ .

(b) That when a line is perpendicular to the plane of projection, its projection on that plane is a point. The lines  $c b$ ,  $c f$ , respectively perpendicular to the vertical and horizontal plane, are projected on those planes in the points  $c'$ ,  $c''$ .

18. It was shown at Figs. 1 and 2, that when the projections of a point are given, the point itself may be deter-

mined, for it is the intersection of the projectors of the point. (See *a*, Fig. 1, and *o*, *p*, Fig. 2.)

19. The solid, at Fig. 3, may also be determined from its projections on the two planes. The surface *b c f g* is the intersection of the projecting plane of *b' c'* with the projecting plane of *c' f'*.

The remaining surfaces are the intersections of the projecting planes of the lines which are the projections of those surfaces.

20. The delineation of the planes of projection, as shown at Figs. 2, 3, has been adopted to convey a clear idea of their relation to each other, and to show how projections are obtained. We shall now show the method employed in practice.

Since objects, whose surfaces lie in different planes, have to be represented upon a sheet of paper, which is but one plane, the vertical plane must be supposed to revolve upon the line of intersection of the planes of projection, until it coincides with the horizontal plane. The vertical plane *A E F B*, Fig. 2, after revolving one fourth of a revolution, as shown by the arcs *H E*, *G F*, will assume the position *A B G H*, which is a continuation of the horizontal plane *A B C D*. Now, after the vertical plane *A E F B* has revolved as described, the vertical projection *o'*, of the point *o*, will assume the position *o''*. In the same manner, the vertical projection *p'*, of the point *p*, will assume the position *p''*; and the line joining *o''* and *p''* will be the vertical projection of a line in space, whose horizontal projection is *o p*.

That *o'' p''* should be the vertical projection of the given line, will, we hope, with a little explanation, be rendered sufficiently obvious. In the first place, each of the lines *o a*, *a o'*, is at right angles to *A B*, the intersection of the planes

of projection; and after the vertical plane has revolved,  $a o''$  is at right angles to  $A B$ . The same observation applies to  $b p''$ . Now, since  $a o''$ ,  $b p''$  are at right angles to  $A B$ , they are at right angles to the horizontal plane  $A B C D$ ,\* and they are, therefore, the projectors of  $o''$ ,  $p''$ , upon that plane; hence,  $o p$ ,  $o'' p''$ , are the horizontal and vertical projections of a line  $o p$  in space (8). Again, the vertical projections of  $o$ ,  $p$ , will be in  $o o''$ ,  $p p''$ , drawn from  $o$ ,  $p$ , at right angles to  $A B$ . Their position is thus determined:—The point  $o$  is the horizontal projection of a point  $o$  in space, which point is elevated above the horizontal plane, a distance equal to  $o o$ . Therefore, set off, from  $a$  in  $A B$ , the distance  $a o''$  equal to  $o o$ . For the same reason, make  $b p''$  equal to  $p p$ , and join  $o'' p''$ ;  $o'' p''$  is the vertical projection of the line.

21. Now, since  $G C D H$  is a representation of the planes of projection, and since this representation is made upon a surface which is but one plane, it is evident, from the foregoing explanation, that the planes will be determined by a line which is the separation of the planes, and previously termed the base line, etc. (3). The line  $A B$ , Fig. 2, determines the planes  $A B C D$ ,  $A B G H$ , independently of the other lines of the figure.

22. In the following problems, we shall thus represent the planes, and the student has only to bear in mind, that one side of the line is the horizontal plane, and that all objects projected upon it are supposed to be viewed as we see objects upon the floor of a room, *i.e.*, by looking *down*; while the other side of the line is the vertical plane, and that all objects projected upon it are supposed to be viewed as we see pictures or other objects upon the walls of a room.

---

\* That is,  $a o''$ ,  $b p''$ , will be at right angles to  $A B C D$  when  $A B G H$  is supposed to be erected or restored to its vertical position by revolving upon  $A B$ .

23. The base line may have any position whatever, with reference to the paper upon which the drawing is to be made; but, for the present, we shall draw it parallel to the upper and lower edges of the paper, and designate it *B L*.

It may be well to point out that *B L*, considered with reference to the vertical plane, is the horizontal plane; while, considered with reference to the horizontal plane, it is the plan of the vertical plane.

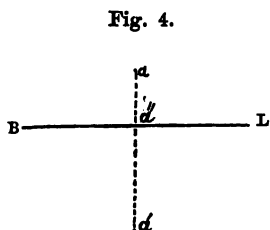
*Obs. 1°.* The base line must always be at right angles to the projectors. Projectors are sometimes called *visual rays*, indicating the direction in which an object is viewed.

*Obs. 2°.* In representing objects orthographically, the eye is supposed to be directly opposite to every point viewed, and the visual rays proceed from these points in parallel straight lines. Thus, in viewing the solid, at Fig. 3, the eye is supposed to be opposite to the points, *a, b, c*, etc. It is this circumstance which constitutes the difference between Orthographic and Perspective Projection—the eye in Perspective being fixed in position. More will be said on this subject when treating of Perspective generally.

#### PROBLEM 1.

*Given the elevation of a point to find its plan.*

Let *a* be the elevation of the given point. From *a* draw a



line perpendicular to *B L*. Now, the plan of *a* will be in this line (8). Let *a'* be its plan. The points *a, a'* are the elevation and plan of a point in space, whose height above the horizontal plane is equal to *a a''*, and whose

distance from the vertical plane is equal to *a' a''*, just as *o', o*,

Fig. 2, are the elevation and plan of a point, whose height above the horizontal plane is  $a o'$ , and whose distance from the vertical plane is  $a o$ ; or, as  $o''$ ,  $o$ , are the elevation and plan of the same point.

### PROBLEM 2.

*Given the elevation of a line at right angles to the vertical plane, to draw its plan.*

Fig. 5.

Since the line is at right angles to the vertical plane, it will be projected on that plane in a point, just as the line  $b c$  is projected on  $A C B L$  in the point  $c$  (see Fig. 3, and also (b), 17). Therefore let  $a$  be its elevation.

The plan of the line is found by letting fall a perpendicular from  $a$ , and making  $a' a''$  equal in length to the given line.

*Obs. 1°.* The line  $a' a''$  is parallel to the horizontal plane, and is projected on that plane in length equal to the original line (15).

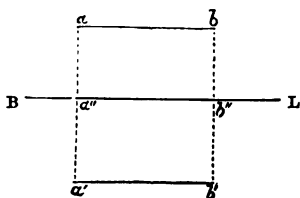
*Obs. 2°.* The line  $a' a''$ , viewed in the direction of its length, will be seen as the point  $a$  in the vertical plane; and  $a$ , viewed from *above*, i.e., at right angles to  $B L$ , will be seen as  $a' a''$ . Thus  $a' a''$  and  $a$  are the plan and elevation of a line at right angles to the vertical plane, and parallel to, and situated above, the horizontal plane at a distance equal to  $a a''$ .

*Obs. 3°.* The line meets the vertical plane in  $a''$  (see 28).

## PROBLEM 3.

*Given the elevation of a line parallel to the two planes of projection, to find its plan.*

Fig. 6.



Since the line is parallel to the two planes, its projections will be parallel to  $BL$  (*(a)*, 17). Therefore, let  $ab$  be its elevation.

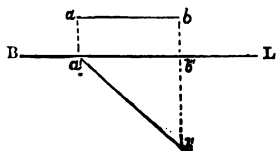
Its plan  $a'b'$  is parallel to  $BL$ , and equal in length to the original line (*(a)*, 17).

The lines  $ab$ ,  $a'b'$  are the projections of a line in space, elevated above the horizontal plane a distance equal to  $aa''$  or  $bb''$ , and removed from the vertical plane a distance equal to  $a'a''$  or  $b'b''$ . (See the projections  $c'd'$ ,  $c''d''$ , Fig. 3, of the line  $cd$ .)

## PROBLEM 4.

*Given the elevation of a line parallel to the horizontal plane, but inclined to the vertical plane of projection, to find its plan.*

Fig. 7.



Let  $ab$  be its elevation. The plan  $a'b'$  shows that the line meets the vertical plane in  $a$ , (*(28)*), and that it is inclined to that plane at an angle  $b'a'b''$ . The plan  $a'b'$  is the real length of the line.

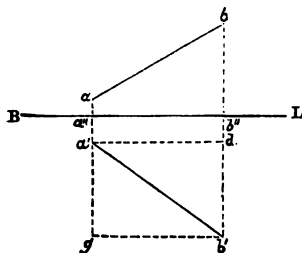
## PROBLEM 5.

*Given the elevation of a line inclined to both planes of projection, to find its plan.*

Let  $ab$  be the elevation.

Taking  $a', b'$ , as the respective distances of points  $a, b$ ,

Fig. 8.



from the vertical plane, the line joining these points will be the plan of  $ab$ .

It will be observed that neither the plan nor elevation expresses the real length of the line, nor its inclination to the planes of projection.

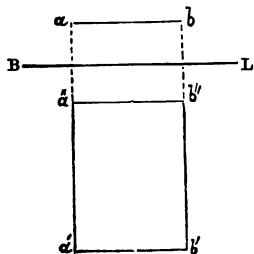
The construction for determining these, will be given in a subsequent problem.

(a) If  $ab$  were the elevation of a line parallel to the vertical plane, either  $a'd$  or  $g'b'$  would be its plan, just as  $a'a'$  or  $b'b''$  expresses the distance of points  $a$  and  $b$  from the vertical plane. That is to say, assuming  $ab$  to be parallel to the vertical plane, its plan will be parallel to  $BL$ . This must be evident, for since every point in the line is the same distance from the vertical plane, every point in the plan must be the same distance from  $BL$ , which may be considered as the plan of the vertical plane (23). Hence, if a line is parallel to the vertical plane, whatever its inclination may be to the ground line, it will be projected upon the horizontal plane in a line parallel to  $BL$ , the ground line.

## PROBLEM 6.

*Given the end elevation of a rectangular surface parallel to the horizontal plane to find its plan.*

Fig. 9.



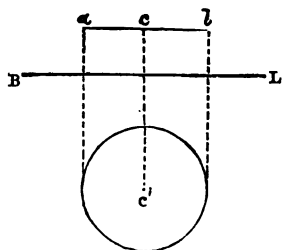
Let  $a b$  be its elevation. The points  $a, b$  represent lines perpendicular to the vertical plane, and the projections of these will be found as in Prob. 2.

Make  $a' a'', b' b''$  equal in length to the given surface, and join  $a' b', a'' b''$ ;  $a' b' b'' a''$  is the plan required.

24. The edge view of a circular surface will be a line equal in length to the diameter of the circle.

Taking  $a b$  as the edge view of a circular surface situated

Fig. 10.



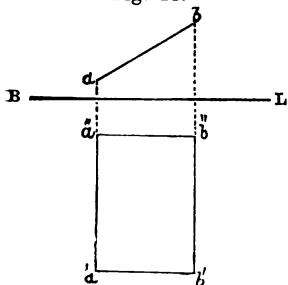
as in the last example, let it be required to draw its plan.

The centre of the circle will be represented at  $c$ , and the required plan will be found by taking any point  $c'$  in the projector from  $c$ , and with this point as a centre, and  $c a$  or  $c b$  as a radius describing a circle.

PROBLEM 7.

*Given the end elevation of a rectangular surface perpendicular to the vertical plane, but inclined to the horizontal plane, to find its plan.*

Fig. 11.



The plan will be obtained as in Prob. 6.

The line  $ab$  and the line beyond it ( $a''b''$  in plan) being parallel to the vertical plane will be parallel to  $BL$ . (See ( $a$ ), Prob. 5.)

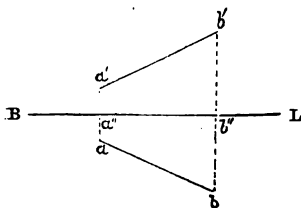
25. The projection of a point from the horizontal to the vertical plane, is the converse of Prob. 1. The vertical projection of  $a'$ , Fig. 4, will be in the perpendicular drawn from  $a'$ , and its position will be found by making  $a''a$  equal to the supposed height or elevation of the point above the horizontal plane.

PROBLEM 8.

*Given the plan of a line to find its elevation.*

Let  $ab$  be the plan of the given line. Draw  $a'a'$ ,  $b'b'$ , at right angles to  $BL$ , making

Fig. 12.



$a'a''$ ,  $b'b''$  equal to the supposed height of  $a$ ,  $b$  above the horizontal plane, and join  $a'b''$ ;  $a'b''$  is the elevation required.

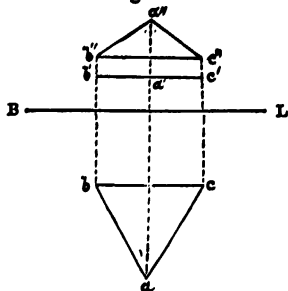
It will be observed, as in Prob. 5, that the line is inclined to both planes of projection.

## PROBLEM 9.

*Given the plan of a triangular surface, to find its elevation.*

Let  $a b c$  be the plan of the given surface. The elevation

Fig. 13.



is  $b' a' c'$ , determined by finding the elevations of points  $a$ ,  $b$ ,  $c$ , and shows that the surface is parallel to the horizontal plane.

Again, let  $a b c$  be the plan, and  $a'' b'' c''$  the elevation. It will be observed, in this case, that the triangle is inclined to the horizontal plane,  $a$  being

elevated above  $b$ ,  $c$ . It is also inclined to the vertical plane; and neither the plan nor elevation gives the real length of the lines  $a b$ ,  $a c$ . The construction for determining this will be given in a subsequent problem.

26. In the preceding problems, the student has been made familiar with the projection of lines and surfaces, perpendicular, parallel, and inclined to, the planes of projection. This combination may be said to embrace the whole theory of Orthographic Projection. In the succeeding problems in Descriptive Geometry, he will only find a modification and extension of the principles already enunciated.

Before commencing the study of these problems, he is recommended to put to himself such questions as the following:—

1. Under what circumstances does the projection of a line become a point? (*/b*), 17.)

2. Under what circumstances is the projection of a line equal to the original line, and under what circumstances is it less? (*/a*), 17, and Prob. 5.)

3. If the projection of a line is parallel to the base line  $BL$ , what is its relation to the other plane of projection? If the projection of a line is parallel to  $BL$ , it will also be parallel to the other plane of projection. The line  $ab$ , Fig. 7, parallel to  $BL$ , is parallel to the horizontal plane. Again, the line  $a'b'$ , Fig. 11, parallel to  $BL$ , is also parallel to the vertical plane. (See also (*a*), Prob. 5.)

4. If both the projections of a line are parallel to  $BL$ , what is the relation of the line to the planes of projection? (Prob. 3.)

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### DESCRIPTIVE GEOMETRY.

27. The intersections of a plane with the planes of projection, are called the *traces* of the plane.

*Note.* In Military Drawings *trace* is another term for *plan*.

28. The intersections of a line with the planes of projection, are called the traces of the line (*Obs.* 3°, Prob. 2, and *a*, Fig. 7).

29. The traces are termed *vertical*, or *horizontal*, as they are referred to the vertical or horizontal plane.

30. When the traces of a plane are given the plane itself is given.

31. When the projections of a line are given, its traces may be found; and, *vice versa*, when the traces are given its projections may be found.

32. The angle between two planes is the angle contained by two straight lines one drawn in each plane, from the same point of their common intersection, and at right angles to it.

33. The angle between two planes is called the *dihedral* angle, and the *profile* angle.

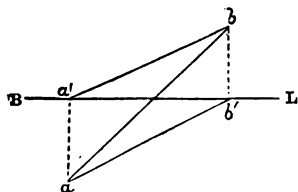
34. The angle between a straight line and a plane, is the angle contained by the straight line and its projection upon the plane.

#### PROBLEM 10.

*Given the traces of a line to find its projections.*

Let  $a, b, *$  be the traces of the given line, *i.e.*, let  $a, b$ , be

Fig. 14.



the points where a line in space meets the planes of projection.

The line joining  $a$  and  $b$  will be the line in space. First, to find its horizontal projection. Draw  $b b'$  at right angles to  $B L$ , and join  $a b'$ ;  $a b'$  is the projection upon the horizontal plane of the line in space.

To find the vertical projection, draw  $a a'$  at right angles to  $B L$ , and join  $a' b$ ;  $a' b$  is the projection upon the vertical plane of the line in space.

Neither the plan nor elevation expresses the real length of the line. To find this will form the subject of the next problem.

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\* In Descriptive Geometry, points in space are usually indicated by capital letters  $A, B, C$ , etc., and their projections by small letters  $a, b, c$ , etc. It is necessary to preserve a consistent notation. Thus, if  $a$  is the projection of a point upon one of the two planes, its projection upon the other plane should be  $a'$ , *i.e.*, the same letter accented. In some works on the subject, the accented letters are confined to the vertical plane. In the following constructions, we shall represent points and lines *first* taken by the letters  $a, b, c$ , etc., whether they are in the horizontal or vertical plane, while the projections of these will be indicated by  $a', b', c'$ , etc.

*Note.* The converse operation of this problem, viz., to determine the traces of a line when its projections are given, would be performed thus:—Let  $a' b'$ ,  $a b$  be the given projections. From  $b'$ , where the horizontal projection meets  $\text{VL}$ , draw  $b' b$  at right angles to  $\text{VL}$ , meeting  $a' b$  in  $b$ ;  $b$  is the vertical trace of the line. Again, from  $a'$ , where the vertical projection meets  $\text{VL}$ , draw  $a' a$ , meeting  $a b$  in  $a$ ;  $a$  is the horizontal trace of the line.\*

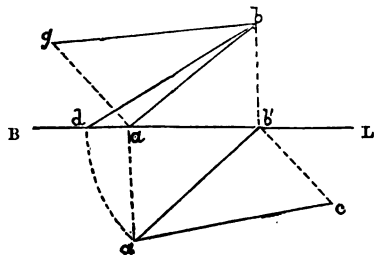
The vertical trace  $b$  shows that  $b'$  is elevated above the horizontal plane a distance equal to  $b b'$ ; while  $a$ , the horizontal trace, shows that  $a'$  is removed from the vertical plane a distance equal to  $a' a$ .

### PROBLEM 11.

*Given the projections of a line to find (1), its length, (2), the angles which it makes with the planes of projection.*

Let  $a b$ ,  $a' b'$  be the projections of the given line.

Fig. 15.



Now, if a vertical plane be conceived to pass through  $a' b'$ , this plane will have  $a' b'$  for its horizontal, and  $b' b$  for its vertical trace. Conceive this plane to revolve upon  $a' b'$  until it coincides

with the horizontal plane. That the student may understand what is meant by the plane revolving upon  $a' b'$ , let him place his triangle with the bevelled edge on  $a' b'$ , and, keeping this edge in contact with the surface of the paper, *turn the triangle down* until it becomes horizontal. It will thus assume

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\* In determining the traces of a line of which neither the horizontal nor vertical projection meets  $\text{VL}$ , produce the projections until they do meet it, and then proceed as explained.

the position of the triangle  $a' b' c$ . To construct this triangle, draw  $b' c$  at right angles to  $a' b'$ , and make it equal to  $b b'$  (for  $b b'$  expresses the height of  $b'$  above the horizontal plane of projection), and join  $a' c$ . The hypotenuse  $a' c$  is the real length of the line.

The line  $a' c$  may be considered as the elevation of  $a' b'$ , when viewed at right angles to the plane, passing through it at right angles to the horizontal plane, *i.e.*, in the direction of the line  $c b'$ .

The construction may also be made in the vertical plane, thus:—Make  $b' d$  equal to  $a' b'$ , and join  $b d$ ;  $b d$  is the real length of the line. The triangle  $b d b'$  represents the vertical plane conceived to pass through  $a' b'$ , after it has been made to coincide with the vertical plane of projection, by being moved through the arc  $a' d$ .

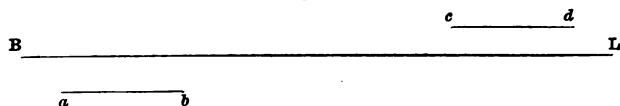
Secondly, to find the angles which the given line makes with the planes of projection. The angle made with the horizontal plane is  $c a' b'$  or  $b d b'$  (34). The angle made with the vertical plane is  $a b g$  which is found thus:—From  $a$ , draw  $a g$  at right angles to  $a b$ , and equal to  $a a'$ . Join  $b g$ .

35. To find the real length of a line whose projections are given, we have this practical rule:—Upon the given horizontal projection construct a right-angled triangle of which the height or perpendicular is equal to the difference of the altitudes of the extremities of the line above the plane of projection.

As referred to the vertical plane, we should make the vertical projection of the line the base of a right-angled triangle of which the perpendicular is equal to the difference of the distances of the extremities of the line from the vertical plane of projection.

36. 1°. When a plane is parallel to either plane of projection, its trace upon the other plane will be a line parallel to the ground line (B L).

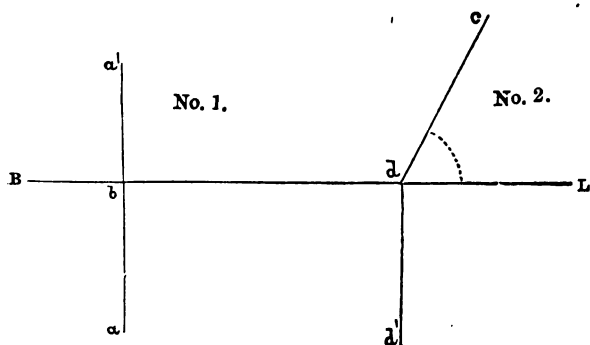
Fig. 16.



The line  $a b$  is the trace of a plane parallel to the vertical plane, while  $c d$  is the trace of a plane parallel to the horizontal plane. In each case the plane will be at right angles to the plane of projection upon which it is represented. The plane  $a b$  is perpendicular to the horizontal plane; while  $c d$  is perpendicular to the vertical plane.

2°. If a plane is perpendicular to either plane of projection, its trace upon the other plane will be a line perpendicular to B L.

Fig. 17.

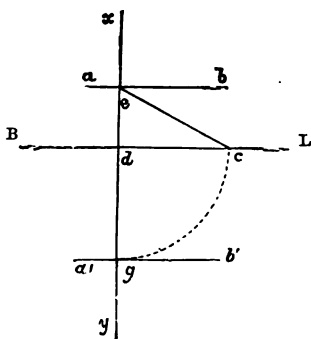


If  $a b$  is a plane perpendicular to the horizontal plane, its vertical trace  $a' b$  will be perpendicular to B L. Again, if

$c d$  is a plane perpendicular to the vertical plane, its horizontal trace  $d d$  will be perpendicular to  $B L$ . In this case, the angle  $c d L$  expresses the inclination of the plane.

3°. If a plane is parallel to  $B L$ , the intersection of the

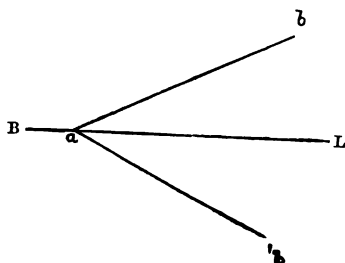
Fig. 18.



planes of projection, it may have two traces  $a b, a' b'$ , each parallel to  $B L$ . The given plane is inclined to the horizontal plane at an angle  $e c d$ , which is found by drawing  $x y$  at right angles to  $B L$ , and making  $d c$  equal to  $d g$ , and joining  $ec$ . The angle  $e c c$  expresses the inclination of the given plane to the vertical plane,

4°. The traces of the plane may have any position with

Fig. 19.



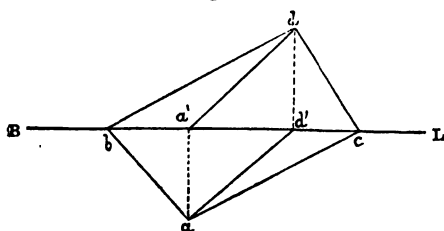
reference to  $B L$ , as  $a b, a' b'$ . In this case the angles which the traces make with  $B L$ , do not express the inclination of the plane to the planes of projection. To find this inclination, see Prob. 13.

*Note.* It will be observed that, in Fig. 19, the traces of the plane meet in a point on  $B L$ , as in No. 2, Fig. 17; but since, in Fig. 17, the horizontal trace  $d' d$  is perpendicular to  $B L$ , the vertical trace  $d c$  expresses the relation of the plane to both planes of projection.

PROBLEM 12.

*Given the traces of two planes, to find the projections of their common intersection.*

Fig. 20.

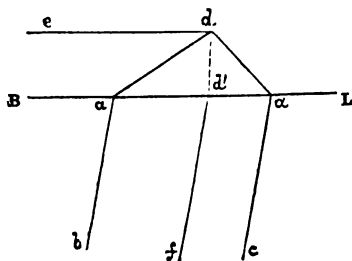


Let  $ab$ ,  $ac$ , be the horizontal traces of the two planes, meeting in  $a$ ; and  $db$ ,  $dc$ , the vertical traces, meeting in  $d$ .

Then, since  $a$ ,  $d$ , are common to the two planes, the line joining these points will be the line in which the planes intersect; and the projections of this line will fulfil the conditions of the problem.

Now, since  $a$ ,  $d$ , are the traces of a line in space, its projections will be found by Prob. 10. Therefore, draw  $dd'$  at right angles to  $BL$ , and join  $ad'$ ;  $ad'$  is the horizontal projection. Again, draw  $aa'$ , at right angles to  $BL$ , and join  $a'd$ ;  $a'd$  is the vertical projection.

Fig. 21.



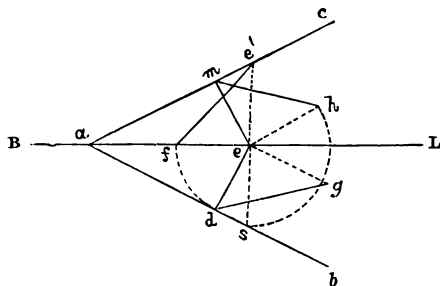
*Obs. 1°.* If the horizontal traces  $ab$ ,  $ac$  (Fig. 21), are parallel, the horizontal projection of their common intersection  $d'f$  will be parallel to  $ab$ ,  $ac$ ; and its vertical projection,  $ed$ , will be parallel to  $BL$ .



### PROBLEM 13.

*Given the traces of a plane, to find the angles which it makes with the planes of projection.*

**Fig. 23.**



Let  $a b, a c$ , be the traces of the given planes. Draw any line,  $d e$ , at right angles to the horizontal trace  $a b$ ; and draw  $e e'$  at right angles to  $b l$ , meeting the vertical trace  $a c$  in  $e'$ .

Then, since  $a c$  intersects the vertical plane (27), the point  $e'$  will be the vertical trace of a line in space whose horizontal projection is  $d e$ .

Find, by Prob. 11, the angle which  $d e$  makes with the horizontal plane. This angle is  $e' f e$  or  $g d e$ . Now,  $d e$  is the projection of  $d g$ ; and since the line and its projection are drawn from the same point  $d$ , which is common to the horizontal plane and the given plane, it follows (32), that the angle  $g d e$  is the angle which the plane makes with the horizontal plane of projection.

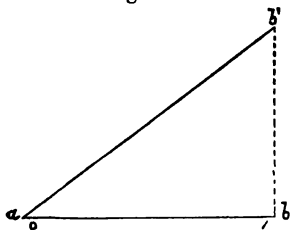
The angle which the given plane makes with the vertical plane of projection, is  $h m e$ , which is found by drawing  $m e$  at right angles to  $a c$ , and constructing the right-angled triangle  $h m e$ , as was done with the triangle  $a b g$ , Fig. 15, *i.e.*, by drawing  $e h$  at right angles to  $e m$ , and making it equal to  $e s$ .

37. When the traces of a line are situated in the traces of a plane, the line is said to *lie* in the plane. The line, joining  $d, e'$ , lies in the plane whose traces are  $a b, a c$ ; and when one projection, as  $d e$ , is given, the other projection may be found.

38. The inclination of a line or plane may be expressed by means of indices.

Thus, if  $a b$  is the plan of a line, and  $o$  (zero) were written

Fig. 24.



at  $a$ , and 1 inch at  $b$ , it would signify that  $b$  is elevated 1 inch above  $a$ ; and taking  $a$  as the point at which the line  $a b$  meets the horizontal plane, then  $b$  is elevated above that plane a distance of 1 inch.

At  $b$ , draw  $b b'$  at right angles to  $a b$ , make it 1 inch, and join  $a b'$ . The angle  $b' a b$  expresses the inclination of the given line to the horizontal plane of projection.

When a drawing is so numbered it is said to be *figured*. This manner of representing heights is of great use, as it enables the draughtsman to make a section or elevation of an object without seeing it.

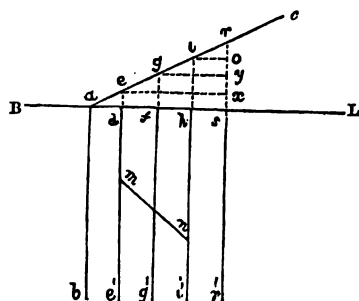
When the elevation of a system of points is required, as in Military Engineering, some above and some below the horizontal plane or the ground, it is necessary to use *positive* and *negative* indices, the positive expressing the elevation of points above the ground upon which a work is constructed; while the negative express distances below the ground. It is usual, however, to dispense with negative quantities, positive being used to express elevations from below the ground. Thus, in military drawings, positive indices are

employed to express the depths of the ditches, as well as the relief\* of the various parts of a work.

39. If a number of equi-distant horizontal lines be drawn in a plane parallel to its horizontal trace, the plane is said to be *contoured*, and the lines so drawn are called *contours* or *horizontals*.

Let  $a b, a c$ , Fig. 25, be the traces of a plane at right angles

Fig. 25.



to the vertical plane of projection, as explained at 2°, No. 2, Fig. 17. If a number of equi-distant lines,  $d e', f g', h i'$  be drawn in the plane parallel to  $a b$ , its horizontal trace, these lines are termed *contours* or *horizontals*.

In Fig. 25, the lines  $d e', f g', h i', s r'$ , represent contours at equal vertical intervals  $s x, x y, y o, o r$ ; and since they are drawn upon a plane having a constant inclination, they are equally distant in plan. This is not always the case, and it is only by clearly understanding the principles of contouring, that objects, represented by their contours, can be recognized. Thus, the plan of the cone (Fig. 61) represented by contours, would be distinguished from the plan of a sphere (the plan of each being represented by a circle) by the concentric circles in the first case being at equal distances apart, while in the second case, they are at the greatest distance apart near the centre, the distance being gradually less as they approach the circumference. The concentric cir-

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\* The height to which works are raised, is called their relief.

cles in each case represent the plans of a number of horizontal sections, taken, as has been explained, at equal vertical distances apart. Again, in representing a hill or an irregular surface by means of contours, the contour lines become nearer to, or farther from, each other, as the hill or surface has a greater or less inclination or steepness.

It is hoped that the following extract will give the student a clear conception of what is meant by *contouring* :—

“Procure a stone somewhat resembling a hill, as may frequently be found, and a box that will just hold it, with a small space around : bed it in clay placed in the bottom of the box, through which there should be a small hole and plug ; fill the box with water stained with Indian ink, and let it off by means of the plug, about a quarter of an inch in depth at several times, allowing sufficient intervals for the fluid to stain the stone in that plane it has fallen to at the last abstraction. These stains will present a series of horizontal lines all over the surface of the stone ; and if we then examine the stone thus prepared, looking down upon the top, we shall see that the steepness and flexure of its sides will be accurately marked with a proportional number of horizontal lines at such variable distances as will very properly express it ; thus, in fact, we obtain a sort of scale of the relative steepness.” \*

Since an irregular surface can only be represented by light and shade, or by the plan of its contours, the latter method is employed by the Engineer in the operations of Topography.

40. It was observed (37) that a line lies in a plane, when the traces of the line are situated in the traces of the plane. A line is also said to *lie* in a plane, when the indices of any

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\* Mr. Burr's Instructions in Practical Surveying and Topographical Plan Drawing.

points in the line are the same as those of the horizontals of the plane passing through those points.

The line  $m n$ , Fig. 25, lies in the given plane if the indices of  $m$ ,  $n$ , are the same as the horizontals  $e d$ ,  $i' h$ , passing through those points. That is to say, if the points  $m$ ,  $n$ , are elevated above the horizontal plane of projection, a distance equal to  $d e$ ,  $h i$ , the line lies in the plane, and not otherwise.

*Obs.* If a plane is given by its contours, its inclination may be found by drawing  $B L$  at right angles to its horizontals (contours), and making upon it an elevation of the plane.

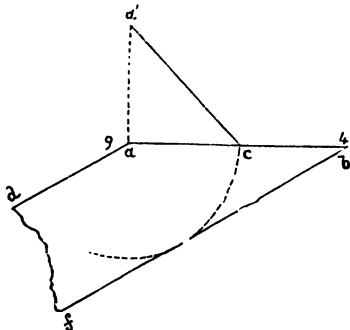
Suppose the plane, Fig. 25, were given by its horizontals  $e d$ ,  $i' h$ , etc.; draw  $B L$  at right angles to the horizontals, and make  $h i$  and  $d e$  (drawn at right angles to  $B L$ ) equal to the height above the horizontal plane of projection expressed by the indices of these horizontals; then the line drawn through the points  $i$ ,  $e$  to  $a$  will give the angle  $c a s$ , which is the inclination of the plane.

41. At Prob. 11, Fig. 15, are given the constructions for finding the real length of a line whose projections are given. This is much facilitated by means of indices. The construction employed at Fig. 24 for finding the inclination of the given line  $a b$ , shows also the real length of the line which is  $a b'$ . To find the length of a line  $a b$ , then, whose indices are 0 and 1, we have simply to draw the elevation of the line at right angles to the given line, making  $b b'$  equal to the difference of the indices of  $a$ ,  $b$ , and joining  $a b'$ . In other words, the real length of a line, given by its figured plan, is the hypotenuse of a right-angled triangle of which the given line is the base, and the difference of its indices the perpendicular (35). For example, if a line in plan measures four inches, and the difference of the indices of its extremities is three inches, the real length of the line is five inches.



problem would be fulfilled by drawing  $BL$  at right angles to the given line, and making on it an elevation of the plane, having the required inclination. For example, suppose  $g'f$ , Fig. 25, were the given line having a given index; draw  $BL$  at right angles to the line, and make  $fg$  equal to the height indicated by the index; then through  $g$  draw  $ca$ , making the angle  $cas$  equal to the required inclination. The rest of the construction will be as shown in the figure.

(a) Suppose the given line  $ab$  were given by its figured  
Fig. 27. plan the indices of its



plan, the indices of its extremities being 9 and 4 feet. Draw  $a' a'$  at right angles to  $a b$ , and make it equal to the difference of the indices  $a, b$ , in this case,  $9-4=5$  feet. (See *Obs.* 1°, Prob. 15.) From  $a'$  draw  $a' c$ , making the angle  $a' c a$  equal to the re-

quired inclination of the plane. With centre  $a$ , and radius  $a c$ , describe an arc, and tangential to it, draw  $f b$ .

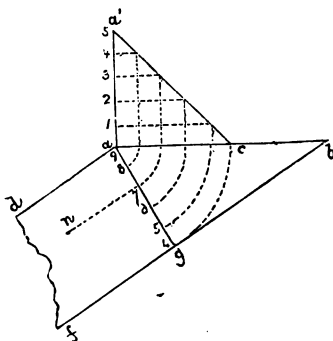
Through  $a$ , draw  $d a$  parallel to  $f b$ ;  $d a, f b$  are two horizontals of the plane, which will be correctly figured by writing 9 upon  $d a$ , and 4 upon  $f b$ .

42. It was observed (See *Obs.* 2°, Prob. 14) that when a line lies in a plane each having the same inclination, the projection of the line will be at right angles to the horizontals of the plane, as  $di$ , Fig. 26. Now,  $di$  may be considered as the scale of the plane  $bdfc$ , as we shall proceed to explain.

Let Fig. 28 be a copy of Fig. 27, the conditions being the

Fig. 28.

same in every respect.



Divide  $a'a'$  into five equal parts, each representing 1 foot; and, through the points of division, draw lines parallel to  $a'b$  to cut  $a'c$ . From the points of division in  $a'c$ , let fall perpendiculars to  $a'b$ , and from  $a$  as a centre, and with each of these points respectively, as a

radius, describe arcs to cut  $ag$ , drawn from  $a$  at right angles to  $da, fb$ . At the points of division in  $ag$  write 4, 5, 6, 7, 8, 9.\* The line  $ag$  thus forms a scale of heights for the plane of which  $da, fb$  are horizontals. Suppose it were required to find the height of any point  $n$  above the horizontal plane of projection. Through  $n$  draw a line parallel to  $da$ , or at right angles to  $ag$ . The division 7, where this line cuts  $ag$ , shows that the given point is elevated above the horizontal plane, a distance of 7 feet.

43. A plane, then, instead of being represented by two or more of its horizontals, may be represented by a line drawn in it at right angles to the horizontals. This line is called the *scale of the plane*; and, that it may represent a scale and not a line, it is usual to make it a double line (one thicker than the other) as in ordinary scale.

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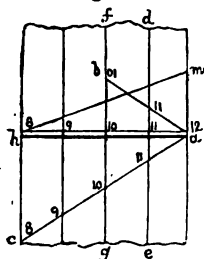
\* It is not necessary to obtain these points, to go through the construction as shown in the figure. It is simply required to divide  $ag$  into five equal parts.

PROBLEM 15.

*To determine the inclination of a plane containing three given points.*

Let  $a, b, c$  be the three given points whose indices are 12, 10, and 8 yards respectively.

Fig. 29.



Join  $a b, a c$ , and divide  $a b$  into 2 equal parts, the difference between the indices  $a, b$ ; also divide  $a c$  into 4 equal parts, the difference between the indices  $a, c$ . Join any two similarly numbered points, say 11 and 11. The line  $d e$ , drawn through these points, will be a

horizontal line, *i.e.*, every point in it will be the same height above the horizontal plane. It will, therefore, be one of the horizontals of the required plane. Other horizontals may be found, as  $f g$ , by drawing lines through the remaining similarly numbered points.

To find the scale of the plane, we have merely to draw  $a h$  at right angles to the horizontals (42). At the points where the horizontals intersect  $a h$ , write the numbers indicating the indices of those horizontals, as 8, 9, etc., and complete the scale by drawing a second line, as shown in the figure.

To determine the inclination of the plane, we have simply to find the elevation of  $a h$ . Now, since  $a$  is elevated 4 yards above  $h$ , make  $a m$  equal 4 yards, and join  $m h$ . The angle  $m h a$  expresses the inclination of the plane. (See 38 and 43.)

*Obs. 1°.* It may be well to point out here, that, in constructing the angle  $m h a$ ,  $a m$  is made 4 yards, not from the



To find the vertical trace, draw  $ag$  parallel to  $fe'$ , meeting  $BL$  in  $g$ ; and through  $a'$ , draw  $a'g'$  parallel to  $BL$ , meeting  $g'g'$ , drawn from  $g$  at right angles to  $BL$ , in  $g'$ . Join  $fg'$  and produce it to  $h$ ;  $fh$  is the vertical trace of the plane.

It will be observed, that  $ag$ , drawn through  $a$ , will be one of the horizontals of the plane, being situated above the horizontal plane of projection at a distance equal to  $a'i$ . Its vertical projection  $a'g'$  will be parallel to  $BL$  (Prob. 4). Now, since this line lies in the plane, and since  $g'$  is its vertical trace, the line, drawn from  $f$  through this point, must be the vertical trace of the required plane. The vertical trace may also be found thus:—Produce  $ab$  to meet  $BL$  in  $m$ , and from  $m$ , draw  $mm'$  at right angles to  $BL$ , meeting  $a'b'$  produced, in  $m'$ . Join  $fm'$ , and produce it to  $h$ ;  $fh$  is the vertical trace.

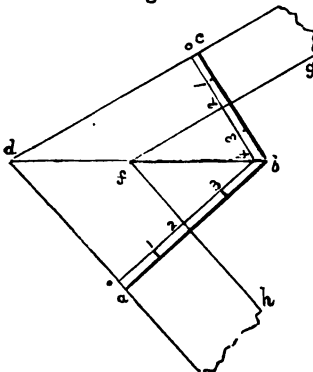
The inclination of the plane will be found as in Prob. 13.

#### PROBLEM 16.

*Given the scales of two planes, to find the intersection of the planes.*

Let  $ab, bc$  be the given scales, each containing 4 divisions,

Fig. 31.



$b$  being elevated 4 yards above  $a$  and  $c$ .

Through any two similarly numbered points, as  $o$  and  $o$ , draw  $ad, cd$  at right angles to the scales, meeting in  $d$ ;  $d$  will be one point in the intersection of the planes. Again, through any other two similarly numbered points, as 2



is  $64^\circ$ ; draw the plan of the planes when lengths of two and three inches on the scales, contain each five divisions.

*Solution.* (1) Make the angle  $a b c$ , Fig. 31,  $64^\circ$ . (2) Make  $a b$ ,  $b c$ , three and two inches respectively, and divide them into five equal parts. The rest of the construction is as shown in the figure.

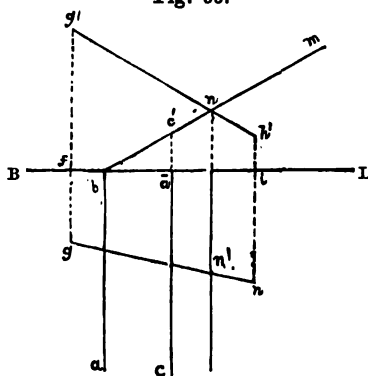
*Note.* It may be well to observe, that the Scales, Fig. 31, need not meet in a point as there shown. They may be inclined to each other without meeting, or they may cross each other.

### PROBLEM 17.

*To find the point of intersection of a line with a plane.*

Let  $a b$ ,  $c d$  be two horizontals of the given plane whose in-

Fig. 33.



indices are 0 and 1 yard; also let  $g h$  be the given line whose indices  $g$ ,  $h$ , are 4 yards and 1 yard respectively. Draw  $B L$  at right angles to the horizontals  $a b$ ,  $c d$ , and upon it make an elevation of the given plane. To do this, we have simply to make  $d c'$  drawn at right angles to  $B L$ , equal to 1 yard

(the index of the horizontal  $c d$  being 1): join  $b c'$  and produce it indefinitely to any point  $m$ .

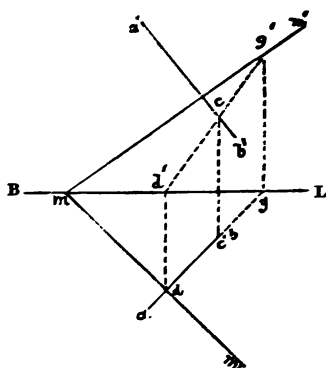
Next, find the elevation of  $g h$  by making  $f g'$ ,  $i h'$ , 4 yards and 1 yard respectively, and joining  $g' h'$ . The point  $n$ , where  $g' h'$  intersects  $b m$ , will be the point of intersection

sought. The horizontal drawn from  $\pi$  intersects the given line in  $\pi'$ , which is the point where  $gh$  intersects the plane.

(a) When the plane is given by its traces, and the line by its plan and elevation.

Let  $m m'$ ,  $m m'$  be the traces of the plane, and  $a b$ ,  $a' b'$ , the plan and elevation of the given line.

Fig. 34.



Conceive a vertical plane to pass through  $a b$ , i.e., conceive a plane to pass through  $a b$  at right angles to the horizontal plane of projection. Produce  $a b$  to meet  $B L$  in  $g$ , and draw  $g g'$  at right angles to  $B L$ , meeting  $m m'$ , in  $g'$ . Then will  $a g$ ,  $g g'$  be the horizontal and vertical traces

of a plane at right angles to the horizontal plane.

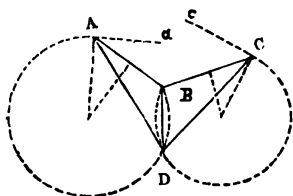
Now, the point where  $a b$  intersects the given plane  $m m'$ ,  $m m'$ , must be common to this plane and to the supposed vertical plane whose traces are  $a g$ ,  $g g'$ . It must, therefore, be in the line in which the planes intersect each other. We have only, then, to find (Prob. 12) the projections of this line.

The traces of the supposed plane cut the traces of the given plane in  $d$  and  $g'$ . Draw  $d d'$  at right angles to  $B L$ , and join  $g' d'$ ;  $g' d'$  is the vertical projection of the intersection of the two planes. The point sought must be in  $g' d'$ ; and since it must also be in  $a' b'$ , it will, therefore, be at  $c$  where the lines intersect each other. Its plan  $c'$  is found by drawing  $c c'$  at right angles to  $B L$ , meeting  $a b$  in  $c'$ .

PROBLEM 18.

*From a station D, the angles between objects A and B, and B and C, were observed to be  $30^\circ$  and  $40^\circ$ , the lines joining A, B, and B, C forming at B an angle of  $130^\circ$ . Find D.*

Let A, B, C represent the positions of the given objects, the angle at B, formed by the lines A B, B C, being  $130^\circ$ .



Now, we want to find a station, or place of observation, such that A, B, may be viewed under an angle of  $30^\circ$ , and B, C under an angle of  $40^\circ$ . Therefore,

by Prob. 15, Prac. Geom., describe upon A B, B C, segments of circles to contain respectively, angles of  $30^\circ$  and  $40^\circ$ . The point D, where these segments intersect each other, will be the point sought. Join D A, D B, D C, and the angles A D B, C D B, will be respectively,  $30^\circ$  and  $40^\circ$ .

Since the whole of the construction is shown in the figure, further explanation will be unnecessary.\*

PROBLEM 19.

*To draw a plane which shall contain a given point, and be parallel to a given plane.*

Let  $m$ ,  $m'$ , be the traces of the given plane; and  $a$ ,  $a'$ ,

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\* For a practical application of this problem, see Col. Jackson's Work on Military Surveying.

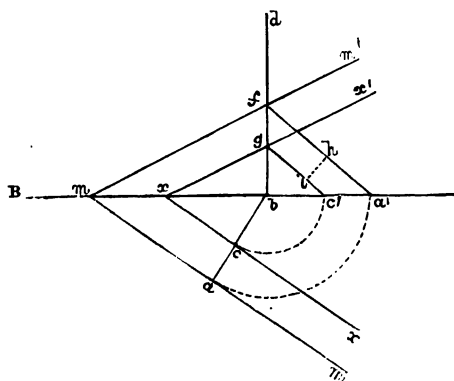


PROBLEM 20.

*Given the traces of two parallel planes, to find the distance between them.*

Let  $m m'$ , and  $x x'$  be the traces of the given

Fig. 37.



planes. Draw  $a b$  at right angles to the horizontal traces of the planes, and  $b d$  at right angles to  $B L$ ;  $a b$ ,  $b d$  will be the traces of a plane at right angles to the horizontal plane of projection.

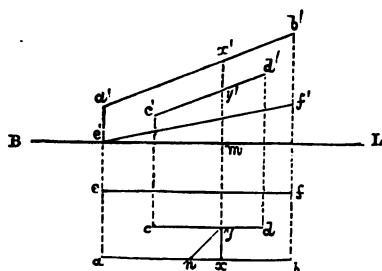
Now, this third plane will cut the given planes in two straight lines, which will be parallel to each other; for if two parallel planes be cut by another plane, their common sections with it are parallel. The plane  $a b$ ,  $b d$  cuts  $m m'$ ,  $m m'$  in the line  $a b$ , and  $x x'$ ,  $x x'$  in  $b c$ . If, therefore, we find (Prob. 13) the angles which the given planes make with the horizontal plane of projection, we shall obtain the distance required. Make  $b c'$ ,  $b a'$  respectively equal to  $b c$ ,  $b a$ , and join  $g c'$ ,  $f a'$ ;  $i h$ , at right angles to these lines, is the distance between the planes.

44. Two lines are not necessarily parallel because their plans are parallel. All lines parallel to the vertical plane, whatever their inclinations may be to the ground line, are

projected upon the horizontal plane in lines parallel to  $BL$ , the ground line (See (a), Prob. 5). If, then, the plans of a number of lines are parallel to  $BL$ , their relation to the vertical plane of projection is known, but not their relation to each other (See Quest. 3, Art. 26). The lines themselves may be actually parallel or inclined to each other.

If the elevations, as well as the plans, of any number of lines are parallel, the lines themselves are parallel.

The line  $ab$  is parallel to  $cd$ , while  $ef$  is not parallel to it. Again, if the difference of the indices  $b$  and  $d$ , is equal to the difference of the indices  $a$  and  $c$ , the lines



are parallel.

It has been shown that two planes are parallel when their traces upon the two

planes of projection are parallel. Two planes are also parallel when equi-distant horizontals in each plane have equi-different indices, increasing in the same direction.

### PROBLEM 21.

*Given the plans and elevations of two parallel lines to find the distance between them.*

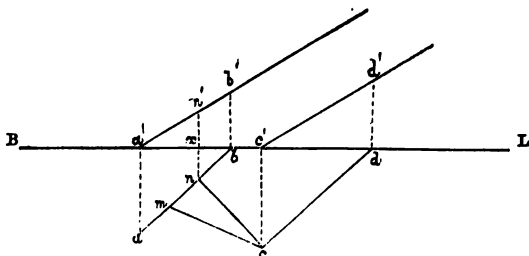
Let  $ab$ ,  $a'b'$ , and  $cd$ ,  $c'd'$  (See last Fig.) be the projections of the given lines.

Draw  $xy$  at right angles to  $ab$ ,  $cd$ , and draw the projector  $x'x'$ , cutting  $a'b'$ ,  $c'd'$  in  $x'$ ,  $y'$ . It will be observed,

that the problem reduces itself to finding the real length of the line of which  $xy$ ,  $x'y'$  are its projections. The difference of the altitudes of  $x$ ,  $y$  above the horizontal plane is expressed by  $x'y'$ . If then, we construct the right-angled triangle  $nxy$ , making  $nx$  equal to  $x'y'$ , we shall obtain  $ny$ , the distance between the lines (See 35, Prob. 11).

(a) Let  $a\ b$ ,  $a'\ b'$ , and  $c\ d$ ,  $c'\ d'$ , Fig. 39, be the projections of the given lines,  $a$ ,  $c$  being their horizontal traces.

**Fig. 39.**



From  $c$  draw  $cn$  at right angles to  $ab$ ,  $cd$ .

Find the altitude of  $n$  above the horizontal plane. This altitude is expressed by  $n'x$ , found by drawing the projector  $n n'$ . Upon  $n c$ , as a base, construct the right-angled triangle  $c m n$ , making  $m n$  equal to  $n'x$ ;  $m c$  is the distance between the lines.

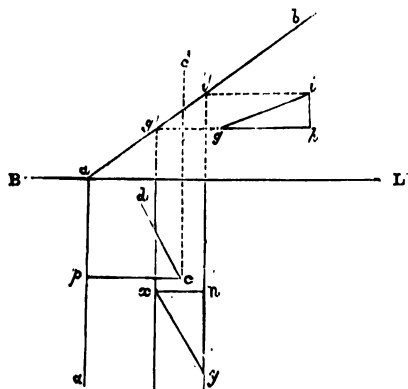
### PROBLEM 22.

*To draw a line through a given point parallel to a given plane, but to have a given inclination.*

**There is a limit to the inclination of the required line; it**

cannot be *greater* than that of the given plane. Let the inclination of the plane be  $35^\circ$ , and that of the line  $25^\circ$ .

Fig. 40.



Let  $a a$ ,  $a b$  be the traces of the given plane,  $a b$  making with  $B L$  an angle of  $35^\circ$ ; and let  $c$ ,  $c'$  be the plan and elevation of the given point.

Construct the right angled triangle  $i g h$ , having the angle at  $g$ ,  $25^\circ$ .\* Draw  $i i'$ ,  $g g'$  parallel to  $B L$ , and

meeting  $a b$  in the points  $i'$ ,  $g'$ . From  $g'$ ,  $i'$ , draw lines parallel to  $a a$ . These lines are two horizontals of the given plane, at a difference of level equal to  $i h$ . Therefore, anywhere between these horizontals, as at  $x y$ , place  $g h$ , the base of the triangle  $i g h$ . Then  $x y$  is a line inclined at  $25^\circ$ , and lying in a plane inclined at  $35^\circ$  (40). It only now remains to draw, through  $c$ , a line  $c d$  parallel to  $x y$ ;  $c d$  is a line inclined at  $25^\circ$ , and parallel to the given plane.

*Obs.* 1°. If the inclination of the required line were the same as that of the given plane, the line would be  $c p$  parallel to  $x n$ , which is drawn in the plane at right angles to the horizontals.

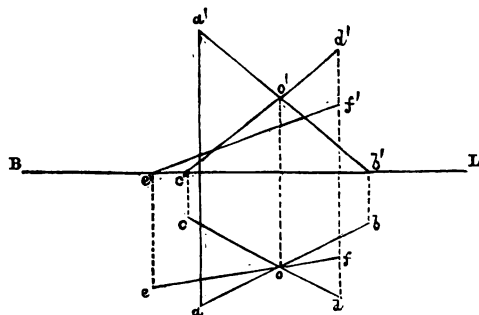
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\* In drawing the angle  $i g h$  equal  $25^\circ$ , it is better to draw from  $i$ , a line  $i g$  making with  $i h$ , the complement of this angle, that is to say, to make the angle  $g i h$ ,  $65^\circ$ . The angle at  $g$  will then be  $25^\circ$ . (See Note, Prob. 26, Fig. 48.)

*Obs. 2°.* If it were required to draw, from the given point, a horizontal line parallel to the given plane, it would simply be necessary to draw, through  $c$ , a line parallel to the horizontals of the plane.

45. Two lines do not necessarily *meet* because their plans *cross*.

Fig. 41.



If the index  $o$ , where the lines  $a b, c d, e f$ , cross, be the same as referred to the three lines, the lines meet, but not otherwise.

Again, if the elevations of the lines cross on the perpendicular drawn from  $o$ , the lines meet. It will be seen that only the lines  $a b, c d$ , meet, as shown by  $o'$ .

### PROBLEM 23.

*To find the angle contained by two lines whose plans are given, or to draw a line from a given point to make a given angle with a given line.*

Let  $a b, b c$  (Fig. 42), be the plans of the given lines,



see the angle contained by the lines at its real magnitude. Conceive the plane to revolve until it coincides with the horizontal plane. In thus revolving, each of the points  $a'$ ,  $c'$ ,  $b'$ , will describe an arc of a circle parallel to the vertical plane. Therefore, with centre  $h$ , and with the points  $a'$ ,  $c'$ ,  $b'$ , as radii, describe arcs cutting  $BL$  in points  $x$ ,  $y$ ,  $z$ . Through  $a$ ,  $b$ ,  $c$ , draw indefinite lines parallel to  $BL$ ; and from points  $x$ ,  $y$ ,  $z$ , draw  $xa''$ ,  $yc''$ ,  $zb''$ , cutting these lines in  $a''$ ,  $c''$ ,  $b''$ . Join  $a''b''$ ,  $b''c''$ ; and the angle  $a''b''c''$  is the real angle contained by the lines.

*Obs.* When a plane has been made to revolve until it coincides with the horizontal plane, it is said to be "constructed," or in other words, "A plane is said to be 'constructed' when the plans of any points, lines, or figures, lying in that plane, are drawn on the supposition that the plane has been turned round on a horizontal till it has been brought parallel to, or coincident with, the plane of projection."

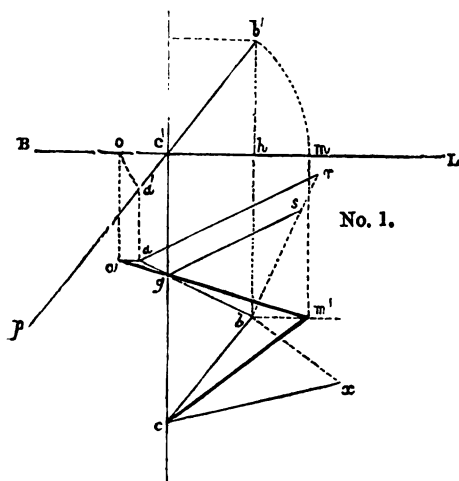
(a) The problem may also be solved thus:—Divide  $ab$  into four parts, and  $bc$  into three parts as explained in the first case. Produce  $ba$ ,  $bc$ , to  $g$ ,  $h'$ , making  $ag$  equal two divisions in  $ab$ ; and  $ch'$  equal three divisions in  $bc$ . Then, since  $g$ ,  $h'$ , are the points where the given lines produced meet the horizontal plane, the line  $gh'$  will be the horizontal trace of the plane containing the lines. Through  $b$ , draw  $bd$  at right angles to  $gh'$ . Find the real length of  $bd$ . This length is shown at  $hb'$ — $mb'$  expressing the difference of the indices  $b$ ,  $d$ . Therefore, make  $db''$  equal to  $hb'$ , and join  $b''g$ ,  $h'b''g$  is the real angle contained by the lines.

(b) The plans of two lines contain an angle of  $80^\circ$ , the

lines being inclined to the horizon at angles of  $50^\circ$  and  $39^\circ$ ; required the real angle contained by the lines.

Let  $a b$ ,  $b c$ , be the given lines inclined at  $50^\circ$  and  $39^\circ$ ,

Fig. 43.

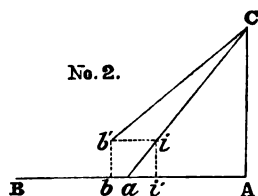


No. 1.

respectively, the angle  $a b c$ , being  $80^\circ$ .

Find a horizontal of the plane containing the two lines. From  $a$ ,  $c$ , draw  $a r$ ,  $c x$ , making with  $a b$ ,  $b c$ , angles of  $50^\circ$  and  $39^\circ$  respectively, and draw  $b r$ ,  $b x$ , at right angles to  $a b$ ,  $b c$ .

Then  $b r$ ,  $b x$ , express the height of  $b$  above  $a$  and  $c$ . Make  $b s$  equal to  $b x$ , and from  $s$  draw  $s g$  parallel to  $a r$ ,



No. 2.

cutting  $a b$  in  $g$ . Then  $g$  will have the same index or altitude above the horizontal plane that  $c$  has; and, therefore,  $c g$  joined will be a horizontal of the plane containing the given lines. Join  $c g$ , and produce it indefinitely. Draw  $B L$  at right angles to  $c g$  produced, cutting it in  $c'$ . Upon  $B L^*$

\*  $B L$  in this case is not the ground line, but a line of level (8).

make an elevation of the plane containing the given lines. Draw the projector  $b b'$ , and make  $b b'$  equal to  $b s$  or  $b x$ . Join  $b' c'$ , and produce it to  $p$ ;  $b' p$  is the elevation of the plane containing the two lines. Project  $a$  to  $a'$ , and "construct" the plane as explained at Fig. 42. When the plane has revolved upon the horizontal  $c g$  till it is coincident with the horizontal plane of projection, the points  $b'$ ,  $a'$ , will assume the positions of  $m$ ,  $o$ . The points  $m'$ ,  $o'$ , are found as in Fig. 42. Join  $m' o'$ ,  $m' c$ , and  $c m' o'$  is the real angle contained by the lines.

*Obs. 1°.* The angle  $b' c' m$  expresses the inclination of the plane in which the given lines  $a b$ ,  $b c$ , are situated.

*Obs. 2°.* The horizontal  $c g$  may be found thus:—Take an indefinite straight line  $B A$  (See No. 2, Fig. 43), and make  $A a$ ,  $A b$  respectively equal to  $a b$ ,  $b c$ . From  $a$  draw  $a c$ , making with  $B A$  an angle of  $50^\circ$ , cutting the perpendicular  $A c$ , in  $c$ . Through  $c$  draw a line making with  $B A$  an angle of  $39^\circ$ ; and from  $b$  draw  $b b'$  at right angles to  $B A$ , cutting this line in  $b'$ . Then, taking  $B A$  as a *line of level*,  $b'$  is elevated above  $a$ , a distance  $b b'$ . From  $b'$ , draw  $b' i$  parallel to  $B A$ , cutting  $a c$  in  $i$ . Then  $b'$ ,  $i$  will have the same index or altitude above the plane of projection. From  $i$  draw  $i i'$  at right angles to  $B A$ . Make  $b' g$  (No. 1) equal to  $i' A$  (No. 2), and join  $c g$ ;  $c g$  is the horizontal required.

*Obs. 3°.* The real angle contained by the lines  $a b$ ,  $b c$  may be found without making an elevation of the plane in which the lines are situated; for since  $g s$ ,  $c x$  are the real lengths of the lines  $g b$ ,  $c b$ , we have simply to describe arcs from  $g$ ,  $c$  with these lines as radii, and to join the point, in which the arcs intersect, to  $g$ ,  $c$ .

(c) To draw a line from a given point to make a given angle with a given line. Let  $ab$ , Fig. 42, be the given line, and  $c$  the given point, the indices of  $a$ ,  $b$ ,  $c$  being the same as there given. Find, as explained, a point in  $ab$  having the same index as  $c$ . We thus obtain the horizontal  $c3$ . Produce  $c3$ , and draw  $BL$  at right angles to it. Make  $x'c'$ ,  $m'b'$ , equal respectively to 3 and 6 ft. Join  $b'c'$ , and produce it to  $h$ . Project  $a$  to  $a'$ , and "construct" the plane containing the given line and point. We thus obtain the line  $a''b''$ , and the point  $c''$ . From  $c''$ , draw a line making with  $a''b''$ , the required angle. Suppose  $a''b''c''$  to be the required angle; then draw  $c''b''$ , making this angle. To find the point  $b''$  in  $ab$ , we have simply to draw  $b''z$  at right angles to  $BL$ , and with centre  $h$ , and radius  $hz$  describe the arc  $b'z$ . Where the projector let fall from  $b'$ , cuts  $ab$ , will be the point  $b$ , which joined to  $c$ , will give the line  $bc$ , making with  $ab$  the required angle. The point  $b$  is also more simply found by drawing a line from  $b''$  parallel to  $BL$  to meet  $ab$ .

46. In drawing a line from  $c$  to make a given angle with  $ab$ , we had first to determine the plane containing the line and point; and then to determine the line and point "constructed," as shown at  $a''b''$  and  $c''$  (See Fig. 42). Having drawn  $c''b''$  to make with  $a''b''$  the required angle, we had next to determine  $b''$  in  $ab$ . In doing this, we performed the *converse* operation to that of "constructing" a plane.

47. To further illustrate this *converse* operation, let  $a''b''$ ,  $b''c''$  be two lines lying in the horizontal plane, and let it be required to find their projection when situated in a plane having a given inclination. Let  $BL$  be the ground line, and  $b'hz$  the angle of inclination of the plane (See Fig. 42).

Draw  $a'x$ ,  $c'y$ , and  $b''z$  at right angles to  $BL$ . With  $h$ ,

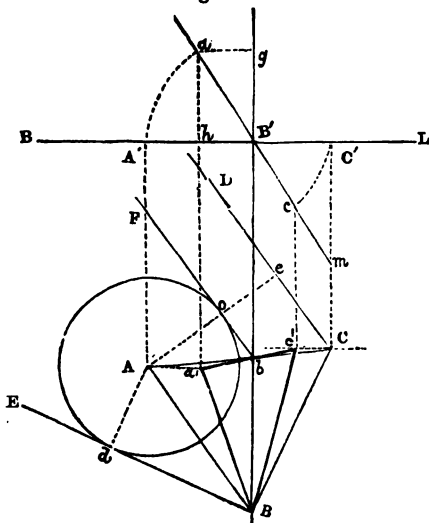
as a centre, and  $h x$ ,  $h y$ ,  $h z$  as radii, describe arcs cutting  $h b'$  in  $a'$ ,  $c'$  and  $b'$ . The points of intersection of the projectors from  $a'$ ,  $c'$ ,  $b'$ , with the lines drawn from  $a''$ ,  $c''$ ,  $b''$  parallel to  $B L$ , will give  $a$ ,  $c$ ,  $b$ . Join  $a b$ ,  $b c$ ; these are the projections of the lines  $a'' b''$ ,  $b'' c''$  when situated in a plane whose inclination is  $b' h z$ .

### PROBLEM 24.

*An equilateral triangle has two of its sides inclined at  $60^\circ$  and  $30^\circ$  to the horizon; draw its plan, and determine the inclination of the plane in which it is situated.*

Let  $\triangle ABC$  be the equilateral triangle. Draw  $CD$ ,  $BE$ ,

**Fig. 44.**



making with  $A C$ ,  $A B$  angles of  $60^\circ$  and  $30^\circ$ . From  $A$  draw  $A d$  at right angles to  $B E$ , and with centre  $A$ , and radius  $A d$ , describe a circle. Parallel to  $C D$  draw  $F b$  tangential to this circle, and cutting  $A C$  in  $b$ . Then since  $A o$  is equal to  $A d$ ,  $B$  and  $b$  will have the same index or altitude above the hori-

zontal plane. Join  $Bb$ , and we have one of the horizontals

of the plane sought. Draw  $B L$  at right angles to  $B b$  produced, and cutting it in  $B'$ . Project the point  $A$  to  $A'$ ; and with centre  $B'$  and radius  $B' A'$  describe an arc. Now, since  $A$  is elevated above  $B, b$ , a distance equal to  $A o$  or  $A d$ , the point  $A'$  must revolve until it is this distance above  $B L$ ; for the plane revolves upon the horizontal  $B b$ , which is represented in elevation by  $B'$ . Therefore, make  $B' g$  equal to  $A o$  or  $A d$ , and draw  $a g$  parallel to  $B L$ , cutting the arc in  $a$ . Join  $a B'$  and produce it to  $m$ ;  $a m$  is the elevation of the plane containing the triangle. Project  $c$  to  $c'$ , and with centre  $B'$ , and radius  $B' c'$  describe an arc, cutting  $a m$  in  $c$ . The plans of points  $a, c$  will be found as in Figs. 42, 43. We thus obtain  $a', c'$ . Join  $a' B, a' c'$ , and  $c' B$ ;  $a' B c'$  is the plan of the triangle, when the sides  $A C, A B$  are inclined at  $60^\circ$  and  $30^\circ$ .

The inclination of the plane in which the triangle is situated, is expressed by the angle  $a B' A'$ .

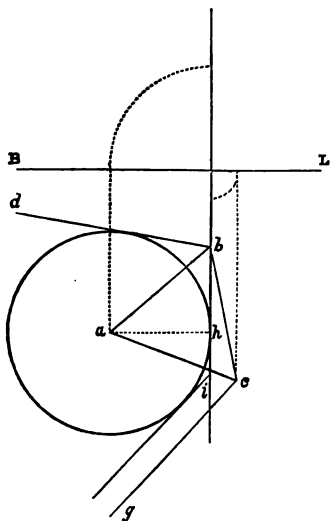
In drawing  $F b$  parallel to  $D C$ , and tangential to the circle, to find the horizontal  $B b$ , the construction is analogous to that employed at Fig. 43, where we made  $b s$  equal to  $b x$ , and drew  $s g$  parallel to  $a r$ .

In Fig. 44,  $A B, A C$  are the real lengths of the lines, and  $d B, e c$  their projections; while in Fig. 43,  $a b, b c$  are projections, and  $a r, c x$  the real lengths of the lines.

*Obs.* 1°. The sum of the angles which the two lines  $A B, A C$  are to make with the horizon, must not be *greater* than

the supplement of the angle at  $A$ . In this case the angle

**Fig. 45.**



at  $A$  is  $60^\circ$ ; therefore, the sum of the angles  $ACD, ABE$  must not be greater than  $180^\circ - 60^\circ = 120^\circ$ , i.e., the sum of the angles must not be greater than  $120^\circ$ . If the sum of the angles were equal to the supplement of the angle contained by the two lines, the plane of the triangle would be vertical.

If in the equilateral triangle  $abc$ , Fig. 45, the angle  $abd$ , expressing the inclination of  $ab$ ,

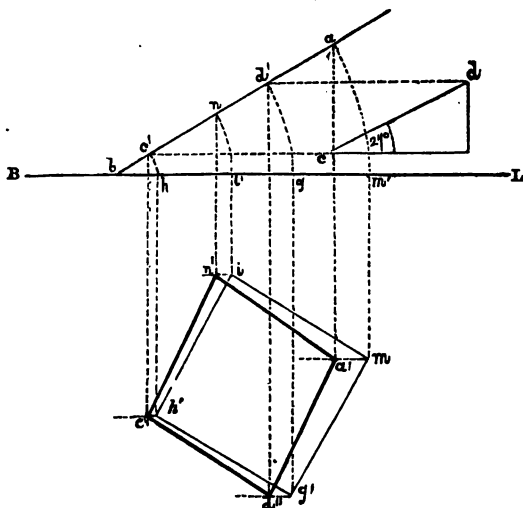
is  $50^\circ$ , and  $a c g$ , expressing the inclination of  $a c$ , is  $70^\circ$ ; then  $50^\circ + 70^\circ = 180^\circ - 60^\circ = 120^\circ$ . In this case the plane of the triangle is vertical, as shown by the construction, and will be projected upon the horizontal of the plane in points  $b, h, i$ .

*Obs. 2<sup>o</sup>.* When the inclinations of two sides of any rectilinear figure are given, the inclination of the plane containing the figure may be determined as explained in the last problem. The position of such figure may also be determined if the inclination of the plane in which it is situated, and that of one side are given, as we shall proceed to show.

## PROBLEM 25.

*The plane of a square is inclined at  $30^\circ$ , and one side at  $27^\circ$ ; draw its plan.*

Fig. 46.



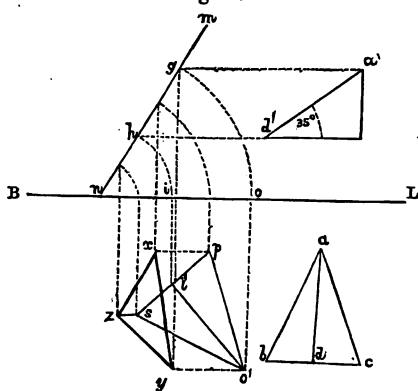
Draw  $a b$  making with  $B L$  an angle of  $30^\circ$ ;  $a b$  is the elevation of the plane containing the given figure. Draw  $c d$ , making with  $B L$  an angle of  $27^\circ$ , and make it equal to the side of the square. Parallel to  $B L$ , draw  $c c'$ ,  $d d'$ , cutting  $a b$  in  $c'$ ,  $d'$ .

Find  $c' d'$  "constructed" as explained at Figs. 42, 43. To do this, we have simply to set off  $c d$  the real length of the line, between the lines drawn from  $h$ ,  $g$  at right angles to  $B L$ . We thus obtain  $h' g'$ . Upon  $h' g'$  describe the square  $g' h' i m$ .

Then by the converse operation to that of "constructing" a plane, we find in  $a b$  the points  $a, n$ , by drawing  $i i', m m'$  at right angles to  $B L$ , and with centre  $b$ , and radii  $b i', b m'$ , describing arcs, cutting  $a b$  in  $n, a$ . The plan of the square is  $a' d' c' n'$ , which is obtained in the same manner as  $a' b c'$ , the plan of the equilateral triangle, Fig. 44.

(a) Draw the plan of an isosceles triangle when its plane is inclined at  $60^\circ$ , and the line drawn from the vertex perpendicular to the base at  $35^\circ$ .

Let  $a b c$  be the given isosceles triangle, of which  $a d$  is the given line drawn from the vertex  $a$  to the base  $b c$ .



is the given line drawn from the vertex  $a$  to the base  $b c$ .

Draw  $m n$  inclined at  $60^\circ$ , and  $a' d'$  equal to  $a d$ , inclined at  $35^\circ$ . Draw  $a' g, d' h$ , and determine  $g h$  "constructed" as was done with  $e' d'$ , Fig. 46. We

thus obtain  $o' i'$ . Through  $i'$ , at right angles to  $o' i'$ , draw a line; and make  $i' p, i' s$ , equal to  $d b, d c$ . Join  $o' p, o' s$ . The projection of the triangle  $o' p s$  is  $x y z$ , found as in the preceding cases.

It will be observed, that the principle involved in the solution of the present problem is analogous to that employed at Prob. 22.



a line  $b'b$  be drawn perpendicular to  $ab$ , the angle  $a'b'b$ , which this line makes with  $a'b'$ , is the complement of the angle which  $a'b'$  makes with  $ab$ ; so that if we determine the angle at  $b'$ , the angle at  $a$  will be the difference between it and an angle of  $90^\circ$ .

If a line is perpendicular to a plane, the *plan* of the line will be at right angles to the horizontals of the plane. Therefore, from  $a$ , draw an indefinite straight line at right angles to  $mn$ , or, what is the same thing, parallel to  $BL$ . From  $a'$  draw  $a'c$  perpendicular to  $mn$ , meeting  $BL$  in  $c$ . Find  $c'$ , the plan of  $c$ , by drawing the projector  $cc'$ . The line  $ac'$ , then, drawn from  $a$  perpendicular to the given plane, makes with  $ab$  an angle  $c'ab$ , which is the complement of the angle which  $ab$  makes with the given plane. Find, by Prob. 23, the real magnitude of this angle. To do this we must determine the plane containing the three points  $a, b, c'$ , Prob. 15. Now, since in the lines  $ab, ac'$ , the points  $b, c'$ , have the same index,  $bc'$  will be one of the horizontals of the required plane; and since the points  $b, c'$  meet the horizontal plane of projection,  $bc'$  will be the horizontal trace of the plane. Therefore, draw  $BL$  at right angles to  $bc'$  produced (See No. 3). Now,  $a$  is elevated above the horizontal plane a distance equal to  $ah$  (No. 2); therefore, draw  $aa''$  at right angles to  $BL$  making  $a'd$  equal to  $ah$ , and join  $a''b''$ ;  $a''b''$  is the elevation of the plane containing the three points. "Construct" this plane (See Fig. 42, and *Obs.* Prob. 23). We thus obtain  $g$ . Draw  $gg'$  at right angles to  $BL$  to meet  $ag'$ , drawn parallel to  $BL$  (No. 3), and join  $bg', c'g'$ . Then  $bg'c'$  is the real angle contained by  $ab, ac'$ ; and since this angle is the complement of the angle required, draw  $g'i$  at right angles to  $bg'$ , and  $c'g'i$  is the angle, which the line  $ab$  makes with the plane whose traces are  $mn, BL$ .

*Obs.* 1°. The angle  $bg'c'$  may be found without going

through the construction shown at No. 3, thus:—Draw  $ao$  at right angles to  $b'o'$ , and find its real length. This is done by constructing a right-angled triangle, making the base equal  $ao$ , and the perpendicular equal to  $a'h$ , the height of  $a$  above  $o$ , and setting off the length of the hypotenuse (along  $ao$  produced) from  $o$  to  $g'$  (35).

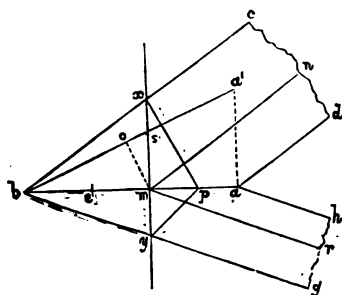
*Obs.* 2°. The line  $ab$  intersects the given plane in the point  $s'$  (Prob. 17); and since the angle, which a straight line makes with a plane, is the angle contained by the straight line and its projection on that plane (34), we have simply to find the real length of  $as'$  as referred, first, to the given plane, and, secondly, to the line  $ab$ .

Now, the real length of  $as'$ , in the first case, is  $s'p$ , found by drawing  $ar$  at right angles to  $ab$ , and making  $ap$  equal to  $xy$ , the difference of the indices, or heights above the plane of projection of  $a$ ,  $s'$ . Again, the real length of  $as'$  in the second case, is  $s'r$ , found by making  $ar$  equal to  $xa'$ , for the reason just given. The angle  $p's'r$  is the angle required.

#### PROBLEM 27.

*To determine the angle between two planes, or, conversely, to draw a plane to make a given angle with a given plane, and to pass through a line in the first.*

Fig. 49.



Let  $abcd$ ,  $abgh$  be the two planes of which  $bc$ ,  $gh$  are the traces; and  $mn$ ,  $mr$ ;  $ad$ ,  $ah$ , horizontals.

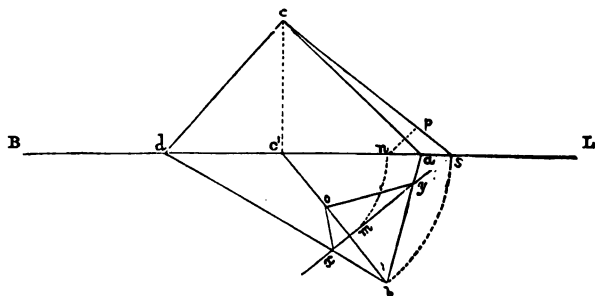
Find the elevation of  $ab$ , the intersection of the planes, i.e., make  $aa'$ , drawn at right angles to  $ab$ , equal to the height of

$a$  above the plane of projection, and join  $a' b$ . Draw  $x y$  at right angles to  $a b$ , and let it be considered as the trace of a plane, cutting the traces of the given planes in  $x, y$ .

Now, if  $x y$  be considered as the trace of a plane at right angles to the horizontal plane, it would cut the given planes in a triangular section. This triangle will be found by making  $m e$  equal to  $m s$  (for  $m$  is elevated above the plane of projection a distance equal to  $m s$ ), and joining  $x e, y e$ . Now, the solution of the problem consists in finding the section of the planes when cut by a third plane at right angles, not to the horizontal plane, but to  $a b$  the planes' intersection. Therefore, draw  $m o$  at right angles to  $a' b$ ;  $m o$  will be the elevation of the plane drawn at right angles to  $a' b$ ; and since this plane contains the lines which measure the angle between the given planes, we have only to "construct"  $m o$  to find this angle. Make  $m p$  equal to  $m o$ , and join  $x p, y p$ ;  $x p y$  is the angle between the planes (32). The angle  $x p y$  is termed the *dihedral angle*, and the *profile angle* of the planes (33).

(a) When the planes are given by their horizontal and vertical traces.

Fig. 50.



Let  $a b, d b$  be the horizontal traces; and  $a c, d c$ , the vertical traces of the planes.

Find by Prob. 12,  $e' b$  the horizontal projection of the



The traces of the plane containing  $a\ b$ ,  $a'\ b'$  will pass through  $a, b'$ ; for a line *lies* in a plane when the traces of the line are situated in the traces of the plane (37). If, then, we can determine another point in either the vertical or horizontal trace of this plane, the line, drawn through these two points to meet  $B\ L$ , will determine such trace.

Let us find a second point in the vertical trace. To do this, draw from  $b'$ , a line  $b'o$  parallel to  $c'\ d'$  meeting  $B\ L$  in  $o$ . From  $d'$  in  $c'\ d'$ , set off  $d'i$  equal to  $b'o$ . Find  $i'$ , the elevation of  $i$ , and make  $o\ o'$ , drawn at right angles to  $B\ L$ , equal to  $ri'$ . Then  $o'$  will be the trace of the line, drawn from  $b'$  parallel to  $c'\ d'$ , and situated in the plane containing the line whose projections are  $a\ b$ ,  $a'\ b'$ . Thus,  $a, o'$  are two points in the trace of the plane passing through the lines joining  $a, b'$ ;  $o', b'$ , in space, and whose projections are  $a\ b$ ,  $a'\ b'$ ;  $o'\ b$ ,  $o\ b'$ .

Now, the line drawn through  $a, o'$ , meeting  $B\ L$  in  $m$ , is the vertical trace of this plane; and the line drawn from  $m$ , through  $b'$ , is the horizontal trace.

To find the trace of the second plane, we have simply to draw, through  $c, d'$ , lines parallel to the traces of the first plane, meeting  $B\ L$  in  $n$ .

*Obs.* The construction shown in the last figure will answer equally to determine a plane which shall contain a given line, and be parallel to another given line. The plane containing the line joining  $a, b'$ , is parallel to the line joining  $c, d'$ .

*Second Case.* When the lines are given by their inclinations instead of their projections upon the two planes; or:—To draw

*a plane inclined at  $52^\circ$  to contain a line inclined at  $36^\circ$ , and to be parallel to another line inclined at  $32^\circ$ .*

For the principle involved in the solution of this question, see Prob. 22.

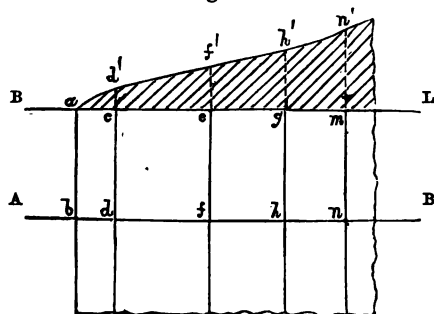
### PROBLEM 29.

*Required the section of a road given by its contours.*

*Def.* In Geometrical Drawings, sections are employed to furnish those details which cannot be represented by a plan or elevation merely, and are designated *sectional planes* and *sectional elevations*. When an object is cut by a vertical plane, i.e., by a plane at right angles to the plane in which the object is situated, the section is termed a *profile*.

Let the contours be  $a b, c d, e f$ , etc., taken at equal verti-

Fig. 52.



cal intervals of 6 feet, the horizontal distances between them being  $\cdot 2, \cdot 5, \cdot 4$ , and  $\cdot 3$  inch respectively.

Draw  $B L$  at right angles to the contours. Then—assuming the scale for the indices to

be  $\frac{1}{160}$ , make  $a d', \cdot 1$  inch (that being the length of 6 feet when drawn to the scale of  $\frac{1}{160}$ ),  $e f', \cdot 2$  inch, and so on to  $m n', \cdot 4$  inch. Through the points  $n', h', f', d',$  and  $a$ , draw

the curve line as shown in the figure. This line will represent the sectional elevation of the given road.

In representing objects in section, it is usual to show the part sectioned by a series of equi-distant parallel lines inclined at  $45^\circ$  to the base line. The direction in which the section is taken, is indicated by a line ( $A B$  in the drawing), called the *section line* or *section plane*.

Since, in the case before us,  $A B$  is at right angles to the horizontals, the section is a profile, and it may be remarked, that a section so taken is the only one which shows the real lengths of the horizontal distances.

*Obs.* Supposing the plan to be drawn to a scale of 1 mile to 1 inch, the distances between the horizontals will be 352, 880, 704, and 528 yards respectively.

#### PROBLEM 30.

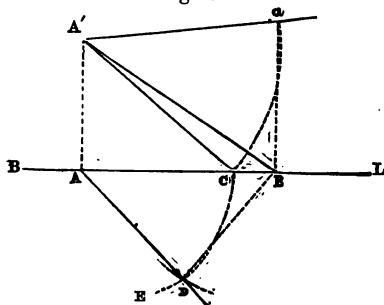
*The observed angle  $BAC$  between two lines  $BA$ ,  $AC$ , whose inclinations to the horizon are  $32^\circ$  and  $37^\circ$  respectively, is  $40^\circ$ ; required the horizontal projection of the angle  $BAC$ .*

*Obs.* 1°. In the operations of surveying, it is usual to consider all prominent objects to be connected by right lines forming triangles, the problem then consisting mainly in measuring the angles of these triangles. In representing the survey upon paper, that is to say, in making a map of a country of which we have made a survey, we have to reproduce these triangles upon a reduced scale, in the same order as that in which they were observed. To do this, the angles of the triangles should be situated in a horizontal plane. If the plane of an angle, that is, if the plane contain-

ing the lines forming an angle, is inclined to the horizon, we have to find its horizontal projection, but not the angle itself (see Prob. 23, Fig. 42). Now, this projection can be found if we know the measured magnitude of the angle, and the inclinations of its two sides to the horizon.

*Solution.* Let  $A, A'$ , be the horizontal and vertical projec-

Fig. 53.



tions of the vertex of the observed angle.

From  $A'$ , draw  $A'C$ ,  $A'B$ , making with  $B L$  angles of  $37^\circ$  and  $32^\circ$ , the inclinations of the sides of the given angle to the horizon.  $A B$  is the horizontal projection of one of

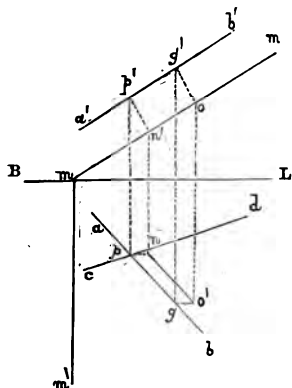
these sides, and it now remains to determine the projection of the other side. If from  $A$  as a centre, and with radius  $A C$ , we describe an arc  $C D E$ , the side  $A' c$  must meet the horizontal plane at some point in this arc. To determine its position, we must find the distance between two points in the lines containing the observed angle. We must, therefore, construct this angle. To do this, draw from  $A'$ ,  $A' a$ , making with  $A' B$  an angle  $a A' B$  of  $40^\circ$ . Next, make  $A' a$  equal to  $A' c$ , and join  $B a$ ;  $B a$  is the distance sought. From centre  $B$ , and with radius  $B a$ , describe an arc cutting  $B D E$  in  $D$ , and join  $A D$ . The line  $A D$  is the horizontal projection of the second side, and the angle  $B A D$  the horizontal angle required.

*Obs. 2°.* The operation performed in the present problem is known by the name of the "reduction of an angle to the horizon."

## PROBLEM 31.

To determine a line which shall be perpendicular to two lines not parallel, nor lying in the same plane.

Fig. 54.

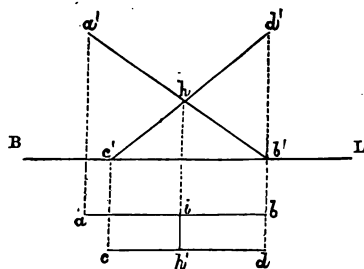


Let  $a'b, c'd$  be the plans of the given lines.

Draw the plane  $m'm'$  to contain one of the lines as  $c'd$ . Now, this plane must be parallel to the line  $a'b$ . Therefore, draw  $a'b'$ , the elevation of  $a'b$ , parallel to  $m'm$ . In  $a'b$  take any point  $g$ , and find its elevation  $g'$ . From  $g'$ , draw  $g'o$  at right angles to  $m'm$ . Find  $g'o'$ , the plan of  $g'o$ . Since  $g'o$  is at right angles to

the plane  $m'm, m'm'$ , its plan  $g'o'$  will be at right angles to the horizontals of the plane. Next, from  $o'$ , draw  $o'n$  parallel to  $a'b$ , meeting  $c'd$  in  $n$ . From  $n$ , draw  $n p$  parallel to  $g'o'$ , meeting  $a'b$  in  $p$ ;  $n p$  is perpendicular to  $a'b, c'd$ .

Fig. 55.



(a) Let  $a'b, c'd$ , the plans of the given lines, be parallel.

Find  $a'b', c'd'$ , the elevations of  $a'b, c'd$ . The point  $h$  in which these lines intersect each other, is the elevation of the required line; its plan is  $h'i$ .



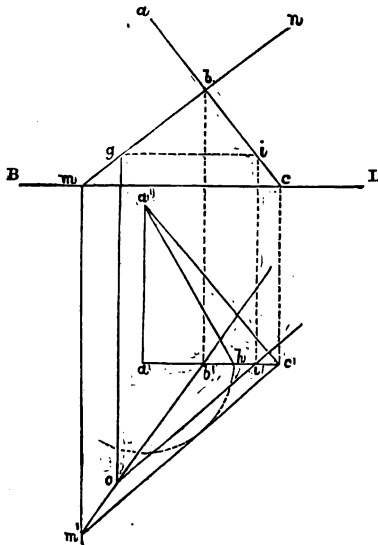
draw  $ag$ ,  $am$  tangential to these arcs, and complete the drawing as shown in the figure.

PROBLEM 34.

*Draw a plane inclined at  $40^\circ$  to the horizon, and a perpendicular to it .5 inch long; through the perpendicular draw a plane inclined at  $60^\circ$  to the horizon, and find by construction the dihedral angle contained by the two planes.*

Let  $m n$ ,  $m m'$  be the traces of the plane inclined at  $40^\circ$ , and  $a b$  the elevation of the line perpendicular to  $m n$ .

Fig. 57.



Produce  $ab$  to meet  $B L$  in  $c$ . Find  $a' c'$ , the plan of  $a c$ ;  $a' c'$  will be at right angles to  $m m'$ , as has been before explained.

Draw  $a'' c'$ , the elevation of  $a' c'$ , by making  $a' a''$  equal to the height of  $a'$  above  $B L$ , and joining  $a'' c'$ . Through  $a''$ , draw  $a'' h$ , making with  $a' c'$  an angle of  $60^\circ$ . With centre  $a'$  and radius  $a' h$ , describe an arc, and tangential to it draw from  $c'$ ,  $c' m'$ ,

meeting  $m m'$  in  $m'$ ;  $m' c'$  will be the horizontal trace of the plane containing  $a' c'$ , which is perpendicular to the given plane.

We have next to determine the line of intersection of the two planes. Now,  $a'c'$  meets or intersects the given plane in  $b'$ ; and since  $a'c'$  lies in the plane whose horizontal trace is  $m'c'$ , it is evident that the point  $b'$  is common to both planes. Therefore, the line joining  $m', b'$  (for  $m'$  is also common to both planes) will be the intersection required. By finding the elevation of  $m'b'$ , the dihedral angle between the planes will be determined as in Prob. 27.

It may be observed, that  $m'b'$  may also be found thus:— Draw  $gi$  parallel to  $BL$ , meeting  $mn, ac$  in  $g$  and  $i$ . Find  $i'$ , the plan of  $i$ . From  $i'$  draw  $i'o$  parallel to  $m'c'$ , meeting  $go$ , drawn parallel to  $mm'$ , in  $o$ ;  $go, i'o$  are two horizontals having the same index, and the line drawn through  $m', o$  is the intersection of the planes (Prob. 16).

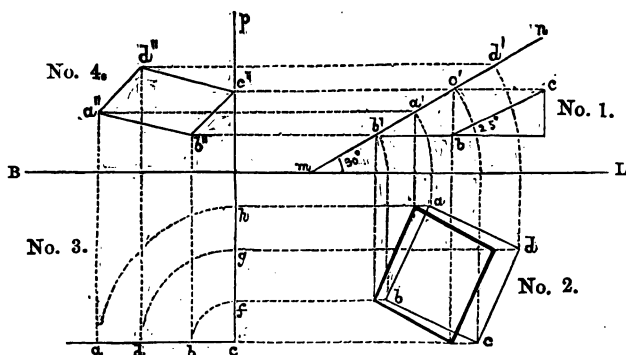
*Obs.* The construction shown in the figure is equally applicable whatever may be the inclination which the given line makes with the given plane, the student has only to bear in mind, that the angle which the line makes with the horizontal plane of projection must not be greater than that which the required plane makes with it (see *Obs.* 1°, Prob. 14).

48. It has been explained (Prob. 25) how to obtain the plan of a square, when the inclination of its plane, and that of one side are given. We shall now give the construction for determining the elevation of such a surface.

#### PROBLEM 35.

*Required the plan and elevation of a square, when its plane is inclined at  $30^\circ$ , and one side at  $25^\circ$ .*

Fig. 58.



Let  $m n$  (No. 1) be the trace of the plane, and  $b' c'$  the elevation of the side of the square, inclined at  $25^\circ$ .

The plan (No. 2.) will be found as in Prob. 25. It now remains to show how to find the elevation.

Taking  $a$  (No. 3) as the plan of a point, and,  $a'$  (No. 1) as the elevation of the same point, the vertical projection of the point will be found by drawing a line from  $a$  (No. 3) at right angles to  $B L$ , to meet a line drawn from  $a'$  (No. 1) parallel to  $B L$ . The point of intersection,  $a''$ , of these lines, is the projection of the point.

For the same reason the projections of each of the points  $b', a', c', d'$  (No. 1) will be in lines drawn from those points parallel to  $B L$ ; and if we know the horizontal distances between the points, we can readily determine the projection of the figure.

Now, the horizontal distance between points  $a', b'$ , is expressed in plan at No. 2. It is the distance between

the lines  $a h$ ,  $b f$ , drawn parallel to  $B L$ . If, then, we set off along  $a c$  (No. 3), drawn parallel to  $B L$ , the length  $a b$  equal to the distance between these lines, and from  $b$  erect a perpendicular to intersect  $b'' b'$ , drawn from  $b'$  parallel to  $B L$ , we obtain  $b''$ , the projection of  $b'$ . Join  $a'' b''$ ;  $a'' b''$  is the projection of  $a' b'$ .

In the same manner, if we make  $a d$ ,  $b c$  equal to the respective horizontal distances between  $a'$ ,  $d'$ , and  $b'$ ,  $c'$  (that is, the distances between the lines  $a h$ ,  $d g$ , and  $b f$ ,  $c e$ ), the intersections of the perpendiculars from  $d$ ,  $c$  with the lines drawn from  $d'$ ,  $c'$ , will give the points,  $d''$ ,  $c''$ , which, joined to  $a''$ ,  $b''$ , and to each other, will give  $a'' b'' c'' d''$ , the elevation of the square when its plane is inclined at  $30^\circ$  and one side at  $25^\circ$ .

The points  $a$ ,  $d$ ,  $b$ ,  $c$ , No. 3, expressing the horizontal distances between the points  $a$ ,  $b$ ,  $c$ ,  $d$ , No. 2, are obtained thus:—Draw  $p c$  at right angles to  $B L$ . From the points  $a$ ,  $b$ ,  $c$ ,  $d$  (No. 2) draw lines parallel to  $B L$ , meeting  $p c$  in  $h$ ,  $g$ ,  $f$ ,  $e$ , and with  $c$  as a centre, and with each of these points respectively, as a radius, describe arcs intersecting  $a c$  in  $a$ ,  $d$ ,  $b$ .

It will be observed, that No. 4 is what No. 1 would look like, when viewed by a spectator looking in the direction of the lines  $a'' a'$ ,  $b'' b'$ , etc. In other words, if the plane  $m n$  revolve upon  $m$  as a hinge, one-fourth of a revolution, we shall obtain No. 4.

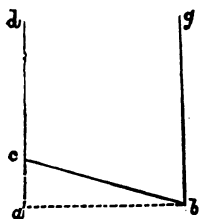
The student will now see the reason why the horizontal distances between the points  $a$ ,  $b$ ,  $c$ ,  $d$ , No. 2, are measured between the lines drawn from those points parallel to  $B L$ , and not from between lines drawn from the points at right angles to  $B L$ .

49. The inclination of a plane is sometimes expressed by a fraction, the numerator of which represents the difference of level between any two points, while the denominator represents the horizontal distance between the same points. In any right-angled triangle, the perpendicular represents the numerator, and the base the denominator of this fraction. Thus, in Fig. 24, if  $ab = bb'$ , i.e., if the base equal the height, the fraction is  $\frac{1}{1}$ , and the angle of inclination is  $45^\circ$ . Again, if  $ab = 3$ , and  $bb' = 1$ , the fraction is  $\frac{1}{3}$ , that is to say, for every unit of level, the horizontals are 3 units apart. In the same way, a plane of  $\frac{1}{4}$  will have its horizontals 4 units apart for every unit of level, and so on. The student will now understand the solution of the following problem.

## PROBLEM 36.

*Two pickets with their upper extremities on the same level and 7 feet apart, stand vertically out of the side of a hill, 6 and 8 feet respectively. Determine the slope of the hill, and give the fraction which represents it, scale  $\frac{1}{8}$ -inch to 1 ft.*

Fig. 59.



Construct the right-angled triangle  $abc$ , making  $ab$ , 7 feet (7 times  $\frac{1}{8}$ -inch); and  $ac$ , 2 feet (2 times  $\frac{1}{8}$ -inch);  $cb$  represents the slope of the hill.

Now, since  $ab = 7$ , and  $ac = 2$ , the fraction is  $\frac{2}{7}$ , the horizontals being 3.5 units apart for every unit of level.

The drawing is completed by making  $cd$ , 6 times, and  $bg$ , 8 times  $\frac{1}{8}$ -inch.

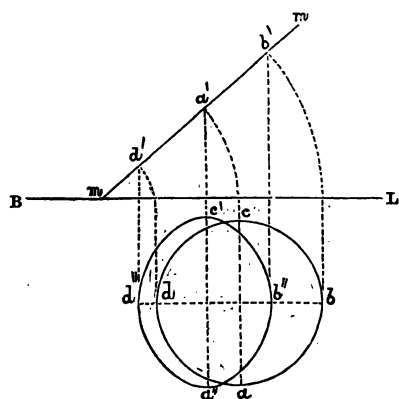
50. If the plane of a circle is horizontal, its plan will be

a circle equal to the original, as shown in Fig. 10. Again, if its plane is vertical, the plan will be a straight line equal in length to the diameter of the circle.

If the plane of the circle, Fig. 10, be vertical, its plan will be a straight line, as  $a b$ , equal to the diameter of the circle. The plan of a circle under any other circumstances, will be an ellipse, as we shall proceed to show.

51. Let it be required to draw the plan of the circle  $a b c d$ , Fig. 60, when its plane is inclined at  $40^\circ$ .

Draw  $m m$ , making with  $B L$  an angle of  $40^\circ$ . Then by the  
Fig. 60.



converse operation to that of "constructing" a plane (see Art. 47, Prob. 23), transfer the points  $d, a, b$ , to  $d', a', b'$ . The diameters  $d b, a c$ , of the circle are represented by  $d' b'$ , and the point  $a'$ . The points  $d'', a'', b''$ , the plans of  $d', a', b'$ , found as in Prob. 23, are three points in

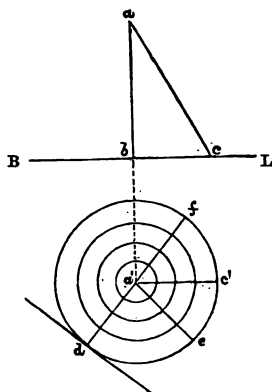
the required ellipse. In the same manner, by taking a further number of points in the circumference of the circle  $a b c d$ , and transferring them to the plane  $m m$ , the curve, described through the plans of these points, will give the

ellipse, which is the projection of the circle when its plane is inclined at  $40^\circ$ .

It will be better, however, to find the projections of  $a c$ ,  $d b$ , the diameters of the circle. Now, since the diameter represented by  $a'$  is horizontal, *i.e.*, since  $a'$  represents the elevation of a line at right angles to the plane of projection, its plan will be equal to  $a c$  (Prob. 2), and is shown at  $a'' c'$ . The plan of the diameter  $d' b'$  will bisect this at right angles, and is shown at  $d'' b''$ .

We have thus found  $a'' c'$ ,  $d'' b''$ , the transverse and conjugate axis of the ellipse, which will be described as in Prob. 25, Prac. Geom.

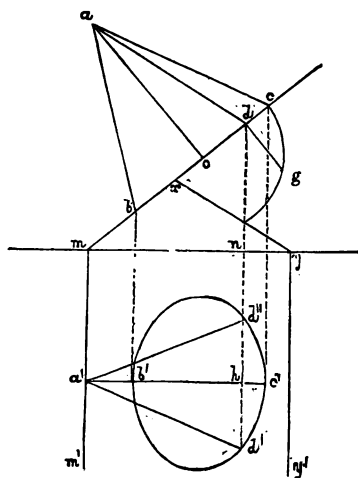
52. If the right-angled triangle  $a b c$ , Fig. 61, revolve upon  $a b$  as an axis, the hypotenuse  $a c$  will generate a conical surface, of which  $d e c' f$  is the plan, the point  $a'$  being the apex. The line  $a c$  is termed a *generator*. Now, if a number of lines, as  $a' d$ ,  $a' e$ ,  $a' f$ , be drawn from the apex to the base of the cone, *i.e.*, to the circumference of the circle, each of these lines may be considered as a generator of the conical surface, and will make with the base of the cone the same angle that  $a c$  makes with



B L.

This fact is independent of the relation of the base of the cone to the plane of projection, that is to say, the angles

which the generators of the cone make with the plane of the base, are the same, whether the base of the cone is horizontal, as in the example before us, or inclined, as shown in Fig. 62.



projected in the point  $a'$  upon  $c' b'$ , the conjugate axis of the ellipse, produced.

Upon the surface of the cone, draw any line as  $a d$ , and let this line represent a section plane (that is to say, a plane cutting the cone), at right angles to the vertical plane (see Prob. 29). The section produced by this plane is represented in plan by the triangle  $a' d' d''$ . The generator  $a c$  is projected in  $a' c'$ . Now, each of the lines  $a' d'$ ,  $a' c'$ ,  $a' d''$ , which may be considered as generators, makes with the base of the cone, *i.e.*, with the plane of the ellipse, the same angle, *viz.*, the angle  $a b c$  or  $a c b$ . In the same manner, if any further number of lines be drawn from the apex  $a'$  to the circumference of the ellipse, each of these lines will make the same angle with the plane of the base of the cone.

The student will now be able to solve the following problem.

### PROBLEM 37.

*To draw a line through a given point to make a given angle with a given plane, but to be parallel to another given plane.*

Let  $a$  (see Fig. 62) be the given point, and  $cm, m'm'$ ;  $xy, y'y'$  the traces of the given planes. Make  $a$  the apex of a right\* cone, of which the generators  $ab, ac$  make with  $cm$  the angle, which the required line is to make with  $cm$ . Through  $a$  draw  $ad$  parallel to  $xy$ , meeting  $cm$  in  $d$ . Determine the horizontal projection of the base of the cone, and the apex  $a$ . Now, taking  $ad$  as a plane cutting the cone, we obtain the points  $d', d''$ , where this plane cuts the base of the cone. Join  $a'd', a'd''$ ; then  $a'd', a'd''$  are two lines drawn through the given point, parallel to the plane  $xy, y'y'$ , and make with  $cm, m'm'$  a given angle.

It will be observed, that the plane  $ad$  cuts the plane  $cm$  in a line, which is shown in plan at  $nd'$ . Now since  $nd'$  is a horizontal of the plane containing  $a'd', a'd''$ , and since this plane is parallel to the given plane  $xy, y'y'$ , the lines themselves are also parallel to this plane; and as they are generators of the conical surface, they make with the plane of the base the given angle  $abc$  or  $acb$ .

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\* A cone is said to be *right* when its axis is at right angles to the plane of its base; and *oblique* when its axis is inclined to the plane of its base.

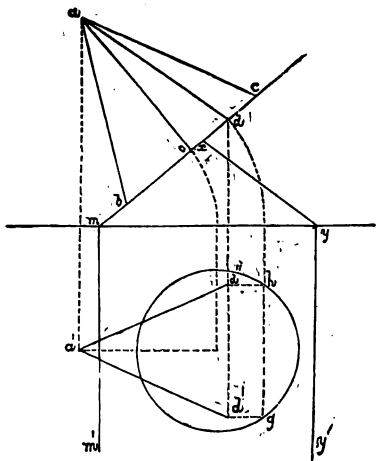
*Obs.* 1°. There is a limit to the conditions of the problem. The angle contained by the two planes (the angle  $mxy$ ), cannot be less than the angle (the angle  $acb$  or  $abc$ ) which the required line is to make with the given plane ( $cm, mm'$ ). If the two angles are *equal*, only *one* line can be drawn; if the angle contained by the planes is *greater*, *two* lines can be drawn, as shown in the figure.

*Obs.* 2°. It is not strictly necessary, in the solution of the problem, to determine the ellipse which is the plan of the base of the cone. To draw an ellipse neatly is by no means an easy operation, and where accuracy can be attained without it, it will be advisable to dispense with it.

(a) The problem, then, may be solved thus:—It will be observed, that the solution consists in finding the points  $d'$ ,  $d''$ . Now, the transverse axis of the ellipse, which is the projection of the base of the cone, will be equal to the original diameter of the circle, as has been explained (51). Thus, the plan of the diameter represented by  $o$ , will be equal in length to the original; and if a number of ordinates be taken in the base of the cone parallel to this diameter, each ordinate will be shown in plan equal in length to the original. Now, we want to find the length of the ordinate represented by  $d$ , that is to say, we want to find the distance between the two points in which the plane  $ad$  cuts the circle which is the base of the cone. With centre  $o$ , and radius  $ob$  or  $oc$ , describe a semi-circle (we have shown only a part of this semi-circle in the figure). From  $d$ , draw  $dg$  at right angles to  $cm$ , meeting the circumference of the semi-circle in  $g$ . From  $h$ , where the vertical from  $d$  intersects

$a'c'$ , set off  $h'd'$ ,  $h'd''$ , each equal to  $d'g$ ;  $d'd''$  is the ordinate expressed by  $d$ . Join  $a'd'$ ,  $a'd''$ ;  $a'd'$ ,  $a'd''$  are the lines which fulfil the conditions of the problem.

Fig. 63.



*Obs. 3°.* By a modification of the construction explained at (a), *Obs. 2°*, the problem may also be solved thus:—"Construct" the plane  $cm$ , Fig. 63, with the base of the cone, and the line in which the plane  $ad$  cuts the base.

This line is shown at  $gh$  in the base of the cone "constructed," and its

projection is  $d'd''$ . The points  $d'$ ,  $d''$ , joined to  $a'$ , the plan of the apex of the cone, give the lines required.

53. If a plane pass over the surface of a right cone, it will cut the base in a straight line, which is a tangent to the circle, representing the plan of the base. If the base of the cone is situated in a horizontal plane, this tangent will be the horizontal trace of a plane, whose inclination is equal to that of the generator of the conical surface. In other words, if any circle whose plane is horizontal, be taken, with its centre, as the plan of a right cone, a tangent to any point of the circle will be the trace of a plane whose inclination is

equal to that, which the generator makes with the plane of the base. If at the point  $d$  (see Fig. 61) a line be drawn a tangent to the circle  $d e c f$ , this line will be the trace of a plane whose inclination is expressed by the angle  $a c b$ . The generator  $a' d$ , at right angles to this tangent, is the line on the conical surface over which the tangent plane passes.

Now, this line  $a' d$  as referred to the tangent plane, is the same as  $d i$ , Fig. 26, Prob. 14, and alluded to at *Obs.* 2°, of the same problem. In fact, the principles just enunciated are employed in the construction shown at Fig. 26, and in several of the subsequent figures.

The line  $b e$ , Fig. 26, tangential to the base of the cone whose generator is  $a c$ , is the horizontal trace of a plane inclined at an angle  $a c d$ , and containing the line  $d h$ .

By a modification of Prob. 14, we are now enabled to solve the following problem.

#### PROBLEM 38.

*Required a plane which shall make a given angle with a given plane, and contain a line not lying in the given plane.*

It will be observed, that, in this problem, we have to do with the given plane, what is done at Prob. 14, Fig. 26, with the horizontal plane.

Let  $m m$ ,  $m m'$  be the traces of the given plane; and  $a b$ ,



## E X A M P L E S.

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*Note.* Those questions marked with an asterisk (\*) are taken from the Reports on the Military Examinations.

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1. A line in plan measures 1 inch, the indices of its extremities being 8 and 3 inches; find the real length of the line, and its inclination to the plane of projection. Scale  $\frac{1}{4}$ -inch to 1 inch.

Make  $a b$ , Fig. 24, 1 inch long. Draw the projector  $b b'$  at right angles to  $a b$ , and make it 5 inches (8-3), the difference of the indices of the extremities of the line, and join  $a b'$ . Since the scale is  $\frac{1}{4}$ -inch to 1 inch,  $b b'$  will be made 5 times  $\frac{1}{4}$ -inch =  $1\frac{1}{4}$  inches, or 1.25 inches.

The real length of the line is  $a b'$ , and its inclination to the plane of projection is expressed by the angle  $b' a b$ .

The same result would be obtained by drawing projectors from  $a$ ,  $b$ , and setting off upon them 3 times and 8 times  $\frac{1}{4}$ -inch respectively. For an explanation of the principle involved in the foregoing solutions, see Art. 38, page 84.

2. (1) Draw a plane inclined at  $50^\circ$ , and in it place a line inclined at  $39^\circ$ . (2) Draw a second line in the plane, at an angle of  $60^\circ$  with the first line, and find the real angle contained by the lines.

For the solution of (1) see Prob. 14, Fig. 26. (2) Assuming  $d h$  as the line inclined at  $39^\circ$ , from any point in it, as  $d$ , draw a line as  $d i$ , making the angle  $h d i$ ,  $60^\circ$ . Then, taking

this line as lying in the plane, and meeting the first line  $d h$  in  $d$ , find the real magnitude of the angle  $h d i$  by Prob. 23.

3. Draw a plane inclined at  $60^\circ$ , and a perpendicular to it 1 inch long; through the extremity of the perpendicular draw a line inclined at  $40^\circ$ , and parallel to the given plane. (Prob. 22.)

4.\* The observed angle from a point  $A$  to two points,  $B, C$ , of which the measured altitudes above the horizon from the same point are  $30^\circ$  and  $35^\circ$  respectively, is  $45^\circ$ . Construct the horizontal angle between the lines  $AB, AC$ . (Prob. 30.)

5.\* From a point  $A$ , the angles between points  $B$  and  $C$ , and  $C$  and  $D$ , were observed to be  $40^\circ$  and  $55^\circ$ , the lines joining  $B, C$ , and  $C, D$ , being 1200 and 1500 yards long respectively, and forming at  $C$  on the side nearest  $A$  an angle of  $155^\circ$ . Find the point  $A$ . Scale 600 yards to 1 inch. (Prob. 18.)

6. Draw the plan of an irregular five-sided figure, when its plane is inclined at  $50^\circ$ , and a line joining any two of its opposite angles at  $38^\circ$ . (Prob. 25, Case (a).)

7. (1) Draw the plan of an isosceles triangle, when its base and one side are inclined at  $40^\circ$  and  $30^\circ$  respectively. (2) Circumscribe the triangle by a circle, and find the plan of the circle. Side of triangle 1.2 inch, and included angle  $50^\circ$ . (Prob. 24. The plan of the circle will be an ellipse.)

8. Upon a line 1 inch long, construct a regular pentagon  $AB C D E$ , and, from its centre  $O$ , draw a line at right angles to one of its sides, as  $C D$ , intersecting it in  $a$ ; draw the plan of the pentagon, when the extremity  $P$ , of a line  $OP$ , 1.4 inch long, and at right angles to the plane of the pentagon, is situated vertically above  $a$ .

Draw  $BL$  at right angles to  $DC$  produced, cutting it in  $F$ .

Parallel to  $DF$ , draw  $OP$ , cutting  $BL$  in  $o'$ , and make  $o'P$  equal to 1.4 inch. Now, the plane in which the pentagon is to lie will revolve upon the horizontal  $DF$ . Therefore, from centre  $F$ , and with radius  $FP$ , describe an arc, cutting  $DF$  produced, in  $G$ . From centre  $G$ , and with radius  $o'P$ , describe an arc. A line drawn from  $F$  tangential to this arc, will be the elevation of the plane containing the pentagon, the plan of which will be found as in Prob. 24.

9. Produce  $oa$  (see last question) to  $b$ , making  $ob$ , 1 inch, and draw the plan of the pentagon, when  $P$  is situated vertically above  $b$ .

10.\* Show, by its traces or by horizontal contours  $\frac{3}{4}$ -inch vertically apart, a plane inclined at  $55^\circ$  to the horizon. In this plane draw a line inclined  $32^\circ$  to the horizon, and another line also in the plane making an angle of  $82^\circ$  with the first line (Prob. 14, Fig. 26, and (c), Prob. 23).

11. The extremities  $c, d$  of two pickets, stand vertically out of the ground, which is horizontal, 8 and 5 feet respectively; express, by means of a fraction, the inclination of the plane in which  $c, d$  are situated, the distance between the pickets being 6 feet. Scale  $\frac{1}{16}$  (Prob. 36).

12.\* The plans of two lines contain an angle of  $80^\circ$ , the lines being inclined to the horizon at angles of  $50^\circ$  and  $35^\circ$ ; determine the real angle contained by the lines and also the inclination of the plane in which they lie. Through the line inclined at  $35^\circ$ , draw a plane (represented as in question 10) making an angle  $75^\circ$  with the plane containing the two lines. (See (b), Fig. 43, Prob. 23, also Prob. 27, Case (b).)

13. Take a line 3 inches long and divide into 7 equal parts. Assuming the line to represent the scale of a plane, draw its elevation, and determine its inclination, when it revolves upon the horizontal drawn through 2, the indices

of the extremities of the line being 0 and 7 inches. Scale for the indices  $\frac{1}{4}$ .

B L in this case will be a line of level, and the question will be solved by reference to Fig. 24, Prob. 16, and Fig. 32.

14.\* A square of 2·3 inches side, the plane of which is inclined at  $47^\circ$ , has one side inclined at  $26^\circ$  to the horizon. Draw the plan of the square (Prob. 25).

15. Two lines  $a b, b c$ , measuring 2·5 and 1·4 inch respectively, contain an angle  $a b c$  of  $110^\circ$ . Taking these lines as the scale of two planes, draw the plan of the planes, when the indices of  $a, b, c$  are 4, 10, and 6 feet. Scale for the indices  $\frac{1}{2}$  (Prob. 16).

16. Find the inclinations of the two planes, and also that of their common intersection, in the last question (Prob. 15).

17.\* Draw the plan of an isosceles triangle having a base of  $1\frac{1}{2}$  inch and sides of  $2\frac{1}{2}$  inches, the triangle being so placed that the base is inclined  $23^\circ$ , and the line joining one end of the base and the centre of the opposite side  $51^\circ$  to the horizon (Prob. 24).

18. Draw the plan of the planes in Question 15, when the indices of  $a, b, c$  are 8, 0, and 6 feet respectively.

*Obs.* In Question 15, the planes form a ridge, while in 18 they form a furrow. (See *Obs.*  $2^\circ$ , Prob. 16.)

19. A line  $a d$ , drawn from the vertex  $a$  of an equilateral triangle  $a b c$ , at right angles to the base  $b c$ , measures in plan 1·4 inches; determine the index of  $a$ , when the index of each point  $b, c$ , is zero. Side of triangle 2 inches.

Construct an equilateral triangle  $a b c$ , and draw  $a d$  at right angles to the base  $b c$ . Next, make  $A B$ , see Fig. 50, Prob. 35, *Prac. Geom.*, equal 1·4 inches, and from  $B$  as a centre, describe an arc with a radius equal to  $a d$ , intersecting  $A c$ , drawn at right angles to  $A B$ , in  $c$ . The line  $A c$

expresses the height of  $a$  above  $b$ ,  $c$ , and, therefore, above the plane of projection, and it will be found to measure 1 inch, that is to say, the index of  $a$  is 1 inch.

20.\* The horizontal projection of a line is 6 feet long and inclined to the axis or line of level at an angle of  $30^\circ$ ; the vertical projection meets the axis at an angle of  $20^\circ$ . Construct the length of the line on a scale of  $\frac{1}{2}$  (Prob. 11).

21.\* The index of one end of a line is 3 feet, of the other 8 feet, and the length of the line is 10 feet. Construct its angle of inclination with the horizon. Scale  $\frac{1}{2}$ -inch to 1 foot. (Fig. 24.)

22. Determine the inclination of a plane containing three points  $a$ ,  $b$ ,  $c$ , whose indices are 7, 5, and 3 feet respectively. Scale  $\frac{1}{2}$ -inch to 1 foot. (Prob. 15.)

23.\* Find the intersection of two planes inclined at  $30^\circ$  and  $54^\circ$  to the horizon, when the projections of their horizontal lines are parallel.

24.\* Draw a line inclined at  $35^\circ$  to the horizon, and through it draw a plane inclined at  $58^\circ$ , and in the latter place a line making an angle of  $30^\circ$  with the first line. (Prob. 14, and (c), Prob. 23.)

25.\* Draw a plane inclined  $42^\circ$  to the horizon, and a perpendicular to it 2 inches long; through the perpendicular draw a plane making an angle of  $55^\circ$  with the horizon, and find by construction the dihedral angle contained by the two planes. (Probs. 34 and 27.)

The second plane containing  $a$ ,  $c$ , Fig. 57, may be determined as in Prob. 14, that is, without finding  $a'$ ,  $c'$ , and the line in which it intersects the first plane, as in Prob. 12.

26.\* The scales of two planes are parallel, the given portion of each is 2 inches long, and one is divided from 10

to 40, the other from 0 to 60. Find the intersection of the planes. See Prob. 16, Fig. 32.

Divide one scale into 3 equal parts, and the other into 6 equal parts, the difference of level of each point of division being 10 units. Find the elevation of the scales as shown at Fig. 32; and the point in which they intersect will be the elevation of the planes' intersection.

27.\* Construct the plan of a square of 2 inches side, resting on a plane inclined at  $50^\circ$  to the horizon, one of the sides being inclined at an angle of  $35^\circ$  to the horizon. (Prob. 25.)

28.\* An equilateral triangle of 3 inches side, rests on one angle, and has the sides adjoining this angle inclined at  $20^\circ$  and  $30^\circ$  respectively to the horizon; construct its plan. (Prob. 24.)

29.\* Draw a plane inclined at  $60^\circ$  to the horizon; in this plane place a straight line inclined at  $46^\circ$  to the horizon. From any point in this line erect a perpendicular to the plane 2 inches long. (Probs. 14 and 34.)

30.\* If two straight lines are parallel their plans are parallel. Prove this, and point out in what case the plan of an angle is equal to the original angle, and whether it is otherwise greater or less. (Art. 44, Fig. 38.)

If the plane of an angle is horizontal, its plan will be equal to the original. If the plane of the angle is inclined, its plan may be either greater or less than the original.

If the line  $b''d$ , Fig. 42, dividing the angle  $a''b''c''$  into two parts, be at right angles to the horizontals of the plane in which the given angle is to lie, the plan of the angle ( $a b c$ ) will be greater than the original angle.

Again, if in the triangle  $a''b''c''$ , the line  $b''d$  be parallel to the horizontals of the plane in which the angle is to lie, the plan of the angle will be less than the original.

31.\* The horizontal distances between 4 contours, cutting a straight road, on a plan drawn on a scale of 10 inches to 1 mile, are 2.5, 5, and 2 inches respectively, and the contours were taken in the field at 25 feet vertical intervals.

Draw the section of the road, showing the horizontal distances on the same scale as that of the plan, and the vertical distances on a scale of 40 feet to an inch (Prob. 29).

32.\* Draw 6 lines, each 7 inches long, and  $\frac{1}{2}$ -inch apart. Suppose these lines to represent the contours of a hill-side at vertical intervals of 5 feet, drawn to a scale of  $\frac{1}{1250}$ . Represent in plan on the hill-side a road 18 feet wide inclined to the horizon at a slope of  $\frac{1}{30}$ .

For the solution of the first part of the question, see Prob. 29. Next, construct a right-angled triangle of which the base is 30 times the perpendicular (49). Place the base of this triangle between the horizontals of the plane (expressing the inclination of the hill) having the same indices. "Construct" the plane with this line. Parallel to the line "constructed," draw a second line at a distance of 18 feet. The rest of the construction will be as shown at Prob. 23. Art. 47.

33. Draw the plan and elevation of an equilateral triangle when its plane is inclined at  $40^\circ$ , and one side at  $36^\circ$  (Prob. 35).

34. A plane is inclined at  $40^\circ$  to the horizon; determine a plane which shall make an angle of  $60^\circ$  with the first plane, and to contain a line inclined to the given plane at an angle of  $45^\circ$  (Prob. 38).

35.\* Draw a plane making an angle of  $42^\circ$  with another inclined at  $50^\circ$  to the horizon, and passing through a line not in the given plane, inclined at  $30^\circ$  to the horizon (Prob. 38).

36\*. A plane inclined at  $60^\circ$  to the horizon makes with

another plane an angle of  $70^\circ$ , the intersection of the two planes being inclined at  $42^\circ$ ; find the inclination, or the scale, of the second plane (Prob. 27, Case (b)).

37. To draw a line through a given point to make an angle of  $60^\circ$  with a plane inclined at  $50^\circ$ , and to be parallel to another plane inclined at  $40^\circ$  (Prob. 37).

38.\* Find the angle contained between two planes, when the angle between their scales is  $65^\circ$ , and lengths of two inches on the scales contain 5 and 7 divisions respectively (Probs. 16, 27).

39.\* (1) An equilateral triangle with a side of 2.2 inches has two of its sides inclined  $25^\circ$  and  $40^\circ$  to the horizon; draw its plan.

(2) Determine the inclination to the horizon of the plane in which the triangle lies.

(3) Draw the plan of the circle circumscribing the triangle (Prob. 24).



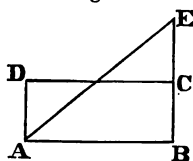


## APPENDIX.

## PROBLEM 1.

*To make a triangle of any given area.*

- (a) Let it be required to make a triangle of 2.5 inches  
Fig. 1.



area. Upon  $AB$  make a rectangle  $ABCD$ , so that  $AB$  is 2.5 inches and  $BC$  1 inch. Then since the area of a rectangle is equal to the length multiplied by the breadth,  $AB \times BC = 2.5 \times 1 = 2.5$  inches.

Produce  $BC$  to  $E$ , making  $CE = CB$ . Join  $AE$ ;  $ABE$  is the triangle required.

The same result would be obtained by producing  $AB$ . (See Prob. 26, Prac. Geom.)

- (b) *Make a triangle whose area is  $\sqrt{3}$  inches.* Construct a right-angled triangle  $ABC$  (see Fig. 43, Prob. 26), making  $AB, BC$  each 1 inch. Then  $AC = \sqrt{2}$ . Next make  $AD = 1$  inch. Then  $DC = \sqrt{3}$ . Again from  $D$  erect a perpendicular to  $DC$  and  $= 2$  inches. Then  $\frac{\sqrt{3} \times 2}{2} = \sqrt{3}$  inches.



- (b) Let  $AB$  be the given base. Upon  $AB$  describe a rectangle equal to twice the area of the triangle.

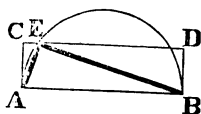
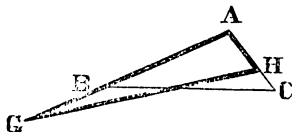


Fig. 3.

Next upon  $AB$  describe a segment of a circle to contain an angle equal to the given angle (see last Figure). At the point  $E$ , where the segment cuts  $CD$ , draw  $EA, EB$ ;  $AEB$  is the triangle required.

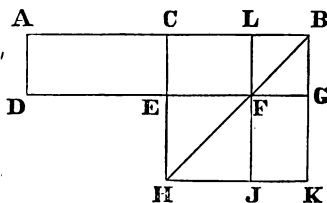
Fig. 4.



- (c) Let the angle at  $A$  be the given vertical angle, and let the area of the required triangle be equal to  $ABC$ ; also let  $DE$  be equal to the sum of the two sides.

Find in  $DE$  a point  $F$ , so that the rectangle  $DF, FE$  is equal to the rectangle contained by  $AB, AC$ . To do this, we must construct the rectangle  $AB, BC$ ; and then by Prob. 32, Prac. Geom., make a square equal to this

Fig. 5.



rectangle. Next take  $AB$  equal to  $DE$ . Bisect  $AB$  in  $C$ . We have now to find a point  $L$  in  $CB$ , such that the rectangle  $AL, LB$  is equal to the difference of the square of  $CB$ , and of the square

just found. This will be done by Prob. 34, Prac. Geom.

In Fig. 4, make  $DF$  equal to  $AL$ . Make  $AG$ ,  $AH$  equal to  $DF$ ,  $FE$ , and join  $GH$ ;  $AGH$  is the triangle required.

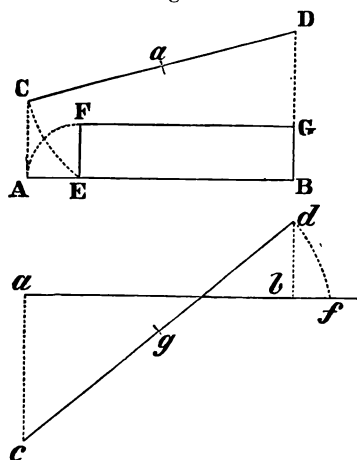
*Proof.* See Euclid II. 5.

*Obs.* 1. The result may be more simply obtained as explained at Fig. 6.

*Thus, let it be required to divide the line  $DE$ , Fig. 4, so that the rectangle contained by its two segments is equal in area to the rectangle under  $AB$ ,  $AC$ .*

Make  $AB$  equal to  $DE$ .

Fig. 6.



From  $A$ ,  $B$ , erect perpendiculars  $AC$ ,  $BD$ , respectively, equal to  $AC$ ,  $AB$  (Fig. 4). Join  $CD$  and bisect it in  $a$ . With centre  $a$  and radius  $aC$ , describe an arc intersecting  $AB$  in  $E$ . From  $D$ , Fig. 4, set off  $DF$  equal to  $BE$ . The rectangle under  $DF$ ,  $FE$  is equal to that under  $AB$ ,  $AC$ ; and  $DE$  is divided as required.

*Obs.* 2. If the given line should be too short, take  $a, b$  equal to the given line. From  $a, b$  erect perpendiculars  $ac$ ,  $bd$  equal to the given lines. Join  $cd$  and bisect it in  $g$ . With centre  $g$  and radius  $gd$ , describe an arc cutting  $ab$  produced in  $f$ . The rectangle contained by  $a, b$ ,  $bf$  is the one required.

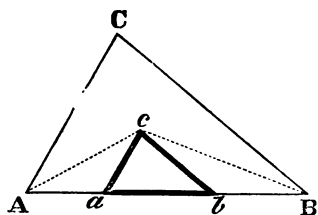
## PROBLEM 4.

*Construct a triangle whose perimeter is  $1\frac{1}{2}$  inches, and its angles as  $2 : 3 :: 4$ .*

*Obs.* The *perimeter* of a triangle is the sum of its sides.

We have first to determine the angles of the triangle. Add 2, 3, and 4 together, and divide 180, the number of degrees in a triangle, by the sum. Thus—

Fig. 7.



$$\frac{180 \times 2}{9} = 40^\circ.$$

$$\frac{180 \times 3}{9} = 60^\circ.$$

$$\frac{180 \times 4}{9} = 80^\circ.$$

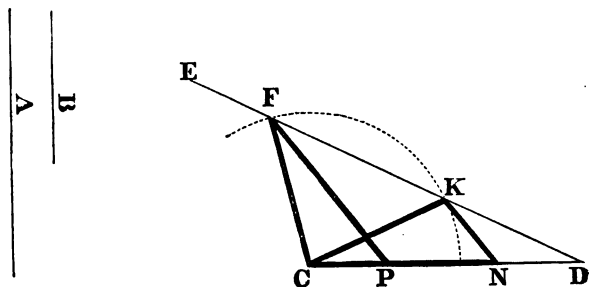
Upon  $AB$ ,  $1\frac{1}{2}$  inches long, describe a triangle  $ABC$ , having its angles  $40^\circ$ ,  $60^\circ$ , and  $80^\circ$ , Prob. 21, Prac. Geom. Bisect the angles at  $A$  and  $B$ , by lines meeting in  $c$ . Draw  $ca$  parallel to  $CB$ , and  $cb$  parallel to  $CA$ ; and  $abc$  is the triangle required. The triangles  $aca$ ,  $bcb$  are isosceles; therefore  $ac$  equals  $ca$ ; and  $bc$ ,  $cb$ .

The proof of the problem is dependent on Euclid I. 29.

## PROBLEM 5.

*One angle of a triangle is  $50^\circ$ , the base  $\cdot 8$  inch, and the sum of the other two sides  $1\cdot 4$  inch; construct the triangle.*

Let  $A, B$ , be the lines expressing the perimeter of the  
Fig. 8.



triangle,  $A$  being 1.4 inch, and  $B$ , .8 inch. Take  $CD$  equal to  $A$ . At the point  $D$ , make the angle  $CDE$  equal to  $25^\circ$ , the half of the given angle. From centre  $C$  and with radius  $B$ , describe an arc of a circle, cutting  $ED$  in  $F$  and  $K$ . At  $K$  make the angle  $NKD$  equal to  $NDK$ . From  $F$  draw  $FP$  parallel to  $KN$ . Either of the triangles  $FCP$  or  $CKN$  fulfils the conditions of the problem. An analysis of the last figure in connection with the observations made thereon, will enable the student to comprehend the construction here employed (see case (*a*) Prob. 3, Appendix).

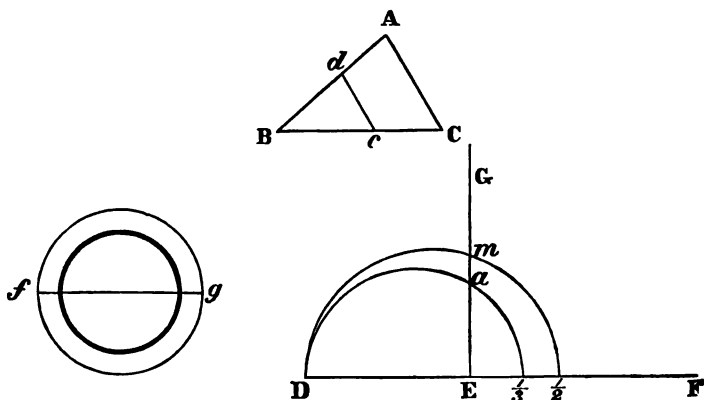
#### PROBLEM 6.

*To draw any figure of an area proportionate to a given similar figure.*

Let it be required to draw a triangle  $\frac{1}{3}$  the area of the triangle  $ABC$ , Fig. 9.

Make  $DE$  equal to  $BC$ . Produce  $DE$  indefinitely towards  $F$ .

Fig 9.



Take  $\frac{1}{3}$  of  $DE$  and set it off towards  $F$ . Upon  $D\frac{1}{3}$ , describe a semi-circle, cutting the perpendicular  $EG$  in  $a$ .

Make  $BC$  equal to  $EA$ , and draw  $cd$  parallel to  $AC$ ;  $BCd$  is a triangle  $\frac{1}{3}$  the area of  $BAC$ .

Next, let it be required to draw a circle  $\frac{1}{2}$  the area of a given circle whose diameter is  $fg$ . Make  $DE$  equal to  $fg$ ; and from  $E$  set off towards  $F$ ,  $E\frac{1}{2}$ . Upon  $D\frac{1}{2}$  describe a semi-circle cutting  $EG$  in  $m$ . With a radius equal to half  $Em$  describe a circle. This circle will be equal in area to half the given circle.

In the same manner a figure may be made, in area, equal to any number of times a given similar figure.

*Obs.* See Prob. 11, Appendix.

#### PROBLEM 7.

*Draw a triangle whose sides shall be as 3 : 4 :: 5, and 4 inches in area.*

The area of a triangle whose sides are actually 3, 4, and 5 inches, is  $\frac{3 \times 4}{2} = 6$  inches.

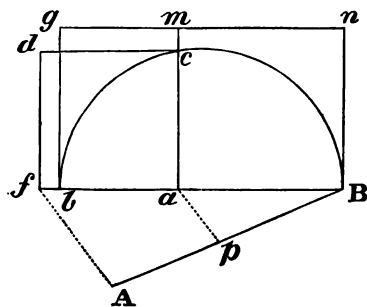
Now the area of the required triangle is  $\frac{2}{3}$  of this. Construct a triangle then whose sides are 3, 4, and 5 inches. Suppose this to be done in the triangle  $EAB$  (Fig. 1, Appendix). Make  $DE$  (see last Fig.) equal to  $AB$ . Take  $\frac{2}{3}$  of  $DE$  and set it off towards  $F$ , and proceed as explained for the triangle  $ABC$ , Fig. 9.

*Obs.* See Prob. 11, Appendix.

### PROBLEM 8.

*To divide a line into two segments so that the squares described on them shall have any given ratio of area to each other.*

Fig. 10.



Let  $AB$  be the given line. Draw  $Bf$  making with  $AB$  any given angle. Set off  $Ba$ ,  $ab$  equal to the given ratio between the squares, taken from any scale of equal parts; in this case  $8.5 : 6.5$ . Upon  $ab$  describe the square  $amnb$ ; and

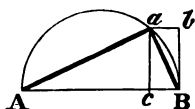
upon  $ab$ , the rectangle  $abgm$ , having the same altitude as the square. Change this rectangle into a square of equal area, Prob. 32, Prac. Geom. Join  $fA$ , and draw  $ap$  parallel to it. The squares described upon  $Ap$ ,  $pB$  will have to each other the given ratio; hence the line  $AB$  is divided, etc.

## PROBLEM 9.

*The hypotenuse of a right-angled triangle is  $\cdot 8$  inch; the perpendicular let fall upon it from the opposite angle  $\cdot 3$  inch; construct the triangle.*

Take  $A B$ ,  $\cdot 8$  inch long, and upon it describe a semi-circle.

Fig. 11.

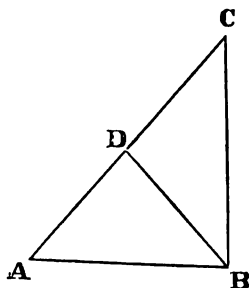


From  $B$  erect the perpendicular  $B b$ ,  $\cdot 3$  inch. Draw  $a b$  parallel to  $A B$ , cutting the semi-circle in  $a$ . Join  $a B$ ,  $a A$ , and  $a A B$  is the triangle required.

## PROBLEM 10.

*Given one angle of a triangle  $50^\circ$ , the base 1 inch, and the sum of the two remaining sides 1.5 inches, to construct it.*

Fig. 12.



Take  $A B$ , Fig. 12, 1 inch, and make the angle  $B A C$ ,  $50^\circ$ . Make  $A C$ , 1.5 inches and join  $C B$ . At  $B$  in  $C B$ , make the angle  $C B D$  equal to  $A C B$ , cutting  $A C$  in  $D$ ;  $A D B$  is the triangle required.

## PROBLEM 11.

*To construct a square which shall have an area in any given ratio to a given square.*

Let the given ratio be as 4 : 9 or  $\frac{4}{9}$ . Make  $DE$  (see Fig. 27, page 28) equal to 4 units, taken from any scale of equal parts. Produce  $DE$  to  $C$ , making  $EC$  equal to 9 units. Bisect  $CD$  in  $F$ , and upon  $CD$ , with  $FC$  as a radius, describe a semi-circle. From  $E$  erect the perpendicular  $EH$ . The square described upon  $EH$  will be to that upon  $ED$  as 9 : 4.

*Proof.* It is shown (page 28) that  $EH = 6$ . Now  $ED^2 = 4^2 = 16$ ; and  $EH^2 = 6^2 = 36$ . Therefore  $16 : 36 :: 4 : 9$ .

*Obs. 1°.* Any similar figures may be treated in the same manner.

*Obs. 2°.* The student will here observe that what is done at Prob. 6, Fig. 9, is more simply accomplished in the present problem.

## PROBLEM 12.

*To construct a triangle, parallelogram, circle, or regular polygon, etc., equal in area to the difference between two given similar triangles, parallelograms, circles, or regular polygons, etc.*

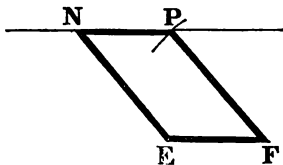
The line  $Ac$ , Fig. 50, found as there explained, will be the base of a similar triangle, the diameter of a circle, etc., equal in area to the difference between the areas of the triangles, circles, polygons, etc., described upon the two lines  $ab$ ,  $cd$ , as there shown for the square.

## PROBLEM 13.

*To make a parallelogram equal to a given triangle in area and perimeter.*

Let  $ABD$  (see last figure) be the given triangle. Make

Fig. 13.



$EF$  equal to half  $AB$ ; parallel to  $EF$ , and at a distance from it equal to the perpendicular let fall from  $D$  upon the base  $AB$ , draw a line. With  $F$  as a centre and radius equal to half  $AC$ , cut this line in  $P$ .

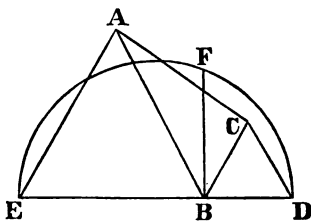
Join  $FP$  and draw  $EN$  parallel to it;  $ENFP$  is the parallelogram required.

## PROBLEM 14.

*To make an equilateral triangle equal to any other triangle.*

Let  $ABC$ , Fig. 14, be the given triangle. Upon  $BC$

Fig. 14.



describe an equilateral triangle  $BCD$ .\* Produce  $DB$ , and from  $A$  draw a parallel to  $BC$ , to cut  $DB$  produced in  $E$ . Find, Fig. 27, a mean proportional to  $DB$ ,  $BE$ . We thus get  $BF$ ;  $BF$  is the side of the equilateral triangle required.

*Obs.* Any multilateral figure, or parallelogram, etc., can, by this problem, be changed into an equilateral triangle,

\* To do this, from  $B$ ,  $C$  as centres, and with radius  $BC$ , describe arcs intersecting in  $D$ . Join  $DB$ ,  $DC$ .

by reducing the given figure to a triangle, and then proceeding as explained.

### PROBLEM 15.

*To construct an irregular five-sided figure A B C D E A under the following conditions :—*

#### *Sides.*

$$A B = 1.2 \text{ inches.}$$

$$B C = .6 \quad ,,$$

$$C D = 1.2 \quad ,,$$

$$D E = .5 \quad ,,$$

#### *Diagonals.*

$$A C = 1.6 \text{ inches.}$$

$$B E = 1.4 \quad ,,$$

$$B D = 1.2 \quad ,,$$

#### *Angles.*

$$A B C = 120^\circ$$

$$B A E = 100^\circ$$

$$B C D = 71^\circ$$

Take A B, see Fig. 14, 1.2 inches long. Make the angle  $A B C = 120^\circ$ . From centre A and with radius 1.6 inches, cut the line last drawn in C. Again, make the angle  $B A E = 100^\circ$ ; and from B with radius 1.4 inches, cut the line last drawn in E. Next, make the angle  $B C D = 71^\circ$ ; and from B, with radius 1.2 inches, cut the line last drawn in D. Join E D; and A B C D E A is the figure required.

*Obs.* The figure might have been constructed under other conditions; for example, the angles of the figure, together with the angles made by the diagonals with the sides of the figure, might have been given; but the manner of proceeding will differ so slightly from that here explained, as not to demand a distinct solution.

### PROBLEM 16.

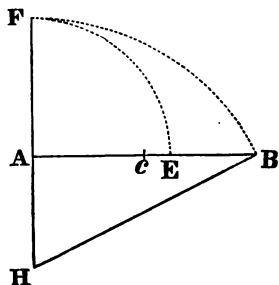
*Draw an equilateral triangle equal in area to the irregular pentagon in the last Problem.*

*Solution.* See *Obs.* Prob. 14, Appendix.

## PROBLEM 17.

*To divide a line  $AB$  into two parts in the point  $E$ , so that the rectangle  $AB, BE$  may be equal to the square of  $AE$ .*

Fig. 15.



Bisect  $AB$  in  $c$ , and through  $A$  draw a line at right angles to  $AB$ ; and make  $AH$  equal to  $AC$ . From centre  $H$  and radius  $HB$  describe an arc cutting the perpendicular through  $A$  in  $F$ . From centre  $A$  and radius  $AF$  describe an arc cutting  $AB$  in  $E$ . Then  $AB, BE = AE^2$  (see Euclid II. 11).

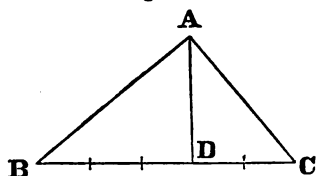
*Obs.* The line  $AB$  is divided into extreme and mean ratio.

## PROBLEM 18.

*To divide a triangle into two parts having the ratio of 3 : 2, by a line drawn from one angle to the opposite side.*

Let  $ABC$  be the given triangle. Divide  $BC$  into 5 equal parts (the sum of  $3 + 2$ ); and at the point  $D$  marking 3 of these, draw  $AD$ , and the triangle is divided as required.

Fig. 16.

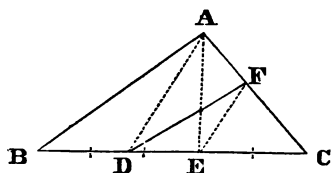


## PROBLEM 19.

*To divide a given triangle into two parts in the ratio of 3 : 2, by a line drawn from a point in one side.*

Let  $A B C$  be the given triangle, and  $D$  the given point.

Fig. 17.



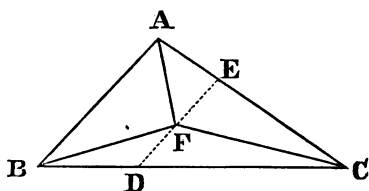
Divide  $B C$  as before; and join  $D A, E A$ . From  $E$  draw  $E F$  parallel to  $A D$ , and join  $F D$ ; then  $F D$  divides the triangle as required.

### PROBLEM 20.

*To divide a triangle into three equal parts by lines drawn from the three angles to meet in a point within the triangle.*

Let  $A B C$  be the given triangle. Take  $B D$  the third part

Fig. 18.

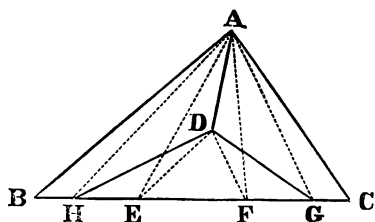


of  $B C$ . From  $D$ , draw  $D E$  parallel to  $A B$ , the side nearer to  $D$ . Bisect  $E D$  in  $F$ , and join  $F$  to the points  $A, B, C$ , and the triangle is divided as required.

### PROBLEM 21.

*To divide a triangle into three equal parts by lines drawn from a point within it.*

Fig. 19.



Let  $A B C$ , Fig. 19, be the given triangle, and  $D$  the given point within it. Divide  $B C$  into three parts, in the points  $E, F$ . Join  $D E, D F, A E, A F$ . Through  $A$  draw  $A H$

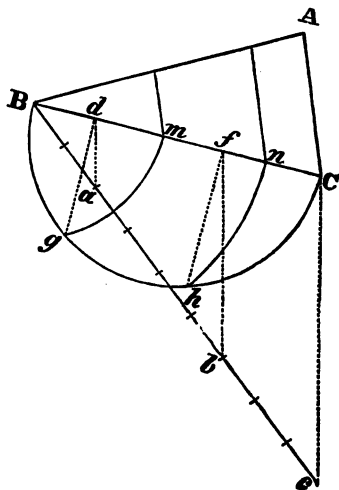
parallel to  $DE$ , and  $AG$  parallel to  $DF$ . Join  $DH$ ,  $DG$ ,  $DA$ ; and the triangle is divided as required.

## PROBLEM 22.

*To divide a triangle into three parts in the ratio of the numbers 2, 4, 3, by lines drawn parallel to one side.*

Let  $ABC$  be the given triangle. Take any line as  $BC$

Fig. 20.



and divide it into 9 equal parts (the sum of  $2 + 4 + 3$ ). Join  $CE$ . Then since  $Ba$  represents 2 parts, and  $ab$ , 4 parts; draw lines from  $a$ ,  $b$  parallel to  $CE$  to meet  $BC$  in  $d$ ,  $f$ . Upon  $BC$  describe a semi-circle, and from  $d$ ,  $f$ , drop perpendiculars  $dg$ ,  $fh$ . With centre  $B$ , and radii  $Bg$ ,  $Bh$ , describe arcs meeting  $BC$  in  $m$ ,  $n$ . From  $m$ ,  $n$  draw

parallels to  $A c$ , and the triangle will be divided as required.

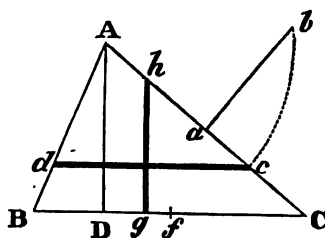
*Obs.* The student will observe that  $B c$  forms a scale of equal parts; and that  $B a$ ,  $a b$ ,  $b c$  represent respectively 2, 4, and 3 of these parts. Now, it is not necessary in the solution of the problem to go through the construction as here shown;  $B a$ ,  $a b$ ,  $b c$  might have been made 2, 4, and 3 parts from any scale of equal parts. This remark applies equally to the previous problems.

### PROBLEM 23.

*To divide a triangle into two equal parts (1) by a line parallel to one of its sides, and (2) by a line perpendicular to one side.*

(1) Let  $A B C$  be the given triangle. Bisect  $A c$  in  $a$ , and

Fig. 21.



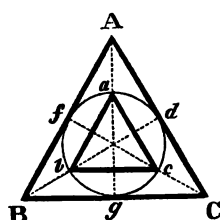
erect the perpendicular  $ab$  equal to half  $A c$ . With centre  $A$  and radius  $Ab$  describe an arc cutting  $A c$  in  $c$ . Draw  $cd$  parallel to  $B c$ ;  $cd$  divides the triangle into two equal parts.

(2) Drop the perpendicular  $AD$ . Bisect  $B c$  in  $f$ , and find the mean proportional (Prob. 12, Prac. Geom.) between  $cd$  and  $cf$ . Set off the length of this mean proportional from  $c$  to  $g$ , and draw  $gh$  perpendicular to  $B c$ ;  $gh$  divides the triangle as required.

## PROBLEM 24.

*To draw an inscribed and a circumscribed equilateral triangle to a given circle.*

Fig. 22.



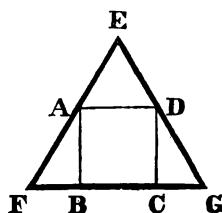
Divide the circumference into six equal parts, by stepping off the radius. Join  $ab$ ,  $bc$ ,  $ca$ ; and  $abc$  is the inscribed triangle.

From centre of circle with radius equal to diameter, describe arcs cutting the lines  $db$ ,  $fc$ ,  $ga$  in  $B$ ,  $C$ ,  $A$ . Join  $AB$ ,  $BC$ ,  $CA$ ; and  $ABC$  is the circumscribed triangle obtained by drawing tangents (Fig. 9, Prac. Geom.) to the circle at the points  $d$ ,  $f$ ,  $g$ .

## PROBLEM 25.

*To describe an equilateral triangle about a square.*

Fig. 23.



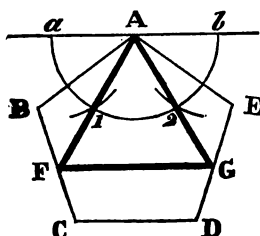
Let  $ABCD$  be the given square. Upon  $AD$  describe an equilateral triangle  $AED$ . Produce  $EA$ ,  $ED$  to meet  $BC$  produced in  $F$ ,  $G$ ; and  $EFG$  is the triangle required.

## PROBLEM 26.

*To inscribe an equilateral triangle in a given regular pentagon.*

Let  $ABCDE$  be the given pentagon. From centre  $A$

Fig. 24.



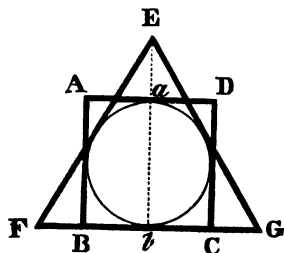
with any convenient radius, describe a semi-circle. From  $a, b$  with radius  $A b$ , describe arcs cutting this semi-circle in  $1, 2$ . From  $A$  through  $1, 2$ , draw lines  $A F, A G$ . Join  $F G$ ; and  $A F G$  is the triangle required.

## PROBLEM 27.

*To circumscribe a circle by a square and an equilateral triangle.*

Draw the diameter  $a b$ . At  $b$  at right angles to  $a b$ , draw

Fig. 25.



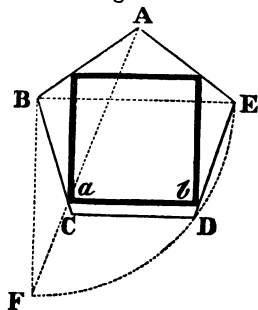
$F G$ . Make  $b B, b c$  each equal to the radius. From  $B, c$  draw parallels to  $a b$ , and make them equal to  $B c$ . Join  $A D$ , and we have the square required.

From the centre of the circle and with radius equal to the diameter, cut  $F G$  in  $F, G$ , and  $a b$  produced in  $E$ . Join  $E F, E G$ , and we have the triangle required.

## PROBLEM 28.

*To inscribe a square in a given regular pentagon.*

Fig. 26.

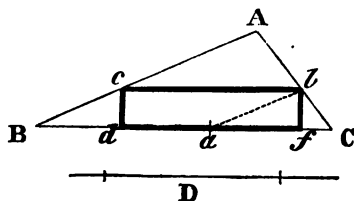


Let  $A B C D E$  be the given pentagon. Join  $B E$  and draw  $B F$  perpendicular to it and equal to  $B E$ . Join  $A F$ , and at the point  $a$  where this line cuts  $B C$ , draw  $a b$  parallel to  $C D$ . Upon  $a b$  construct the square.

## PROBLEM 29.

*In a given triangle to inscribe a rectangle having one of its sides equal to a given line.*

Fig. 27.



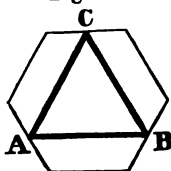
Let  $A B C$  be the given triangle, and  $D$ , the given line.

Make  $B a$  equal to  $D$ . Draw  $a b$  parallel to  $A B$ , and  $b c$  parallel to  $B C$ . From  $b, c$  let fall perpendiculars  $c d, b f$ ; and  $c d f b$  is the rectangle required.

## PROBLEM 30.

*To inscribe an equilateral triangle in a regular hexagon, so that its sides shall be parallel to three sides of the hexagon.*

Fig. 28.

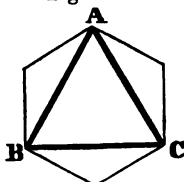


Bisect the sides A, B, C of the hexagon in A, B, C. Join these points and we have the triangle required.

## PROBLEM 31.

*To inscribe the largest equilateral triangle in a regular hexagon.*

Fig. 29.

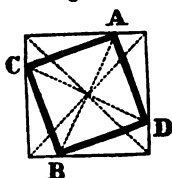


Join any three alternate angles, as A, B, C, and we have the triangle required.

## PROBLEM 32.

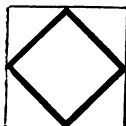
*In a given square to inscribe a square, one corner of which shall touch the given square in a given point.*

Fig. 30.



Let A be the point in the given square. From A through centre of given square, draw A B. At right angles to A B and through centre of given square, draw C D. Join A C B D, and we have the square required.

Fig. 31.



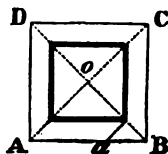
When the corners of the inscribed square are in the centres of the sides of the circumscribing square, the inscribed square is the least possible, as shown at Fig. 31.

## PROBLEM 33.

*Make a square of  $\cdot 6$  inch side, and construct a square concentric with it, having a side of  $\cdot 4$  inch.*

Make  $AB$  equal to  $\cdot 6$  inch, and upon it construct a square. Draw the diagonals. Make

Fig. 32.



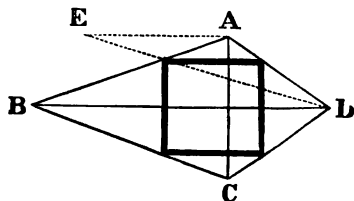
$Aa$  equal  $\cdot 4$  inch, and from  $a$  draw a line parallel to  $AC$ , cutting  $DB$ . From the point thus found, draw a line parallel to  $AB$  until it cuts  $AC$ , and upon this line make the square. (See Fig. 27, Appendix.)

## PROBLEM 34.

*To inscribe a square in a given trapezium  $ABCD$ , when  $AD$  is equal to  $DC$ , and  $AB$  to  $BC$ .*

Draw the diagonals  $AC$ ,  $BD$ . From  $A$  draw  $AE$  parallel

Fig. 33.



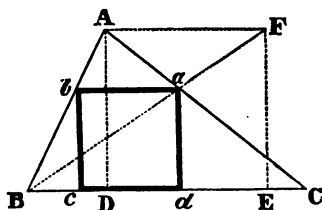
to  $BD$  and equal to  $AC$ . Join  $ED$ . The rest of the construction will be sufficiently obvious without further explanation. (See Fig. 26, Appendix.)

## PROBLEM 35.

*To inscribe a square in a given triangle.*

Let  $ABC$  be the given triangle. Let fall the per-

Fig. 34.



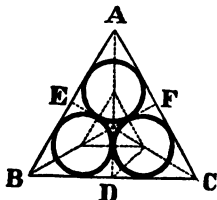
pendicular  $AD$ . Upon  $AD$  construct the square  $ADef$ . Join  $Bf$  cutting  $AC$  in  $a$ . Draw  $ab$  parallel to  $BC$ , and upon it construct the square  $abcd$ .

## PROBLEM 36.

*To inscribe three equal circles in an equilateral triangle, each to touch the other two, and also two sides of the triangle.*

Let  $ABC$  be the given triangle. From the points  $A, B, C$ ,

Fig. 35.



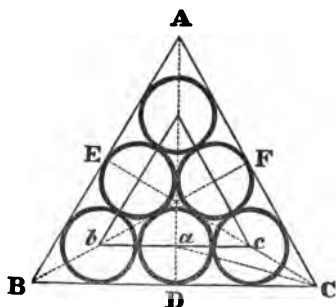
draw perpendiculars  $AD, BF, CE$  to the opposite sides. Bisect the angle  $ADC$ , and from the point where the bisecting line cuts  $CE$ , draw a line parallel to  $BC$ . Upon this line draw the inner equilateral triangle. The angular points of this triangle will be the centres of the required circles, the radius of which will equal half the side.

circles, the radius of which will equal half the side.

### PROBLEM 37.

*To inscribe six equal circles in an equilateral triangle.*

Let  $\triangle ABC$  be the given triangle. Draw the perpendiculars  $AD$ , etc., as in Fig. 36.



diculars  $AD$ , etc., as in the last figure. Bisect the angle  $ECB$  by  $ca$ . Through  $a$  draw  $bc$  parallel to  $BC$ , and upon it construct the inner equilateral triangle.

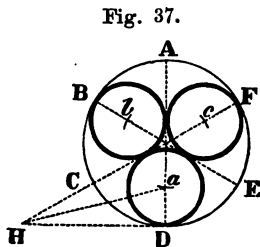
The rest of the construction will need no explanation.

*Obs.* The construction will also serve to show how to inscribe *three* equal circles in an equilateral triangle, which shall touch each other, and one side only of the triangle, see the circles at D, E, F.

### PROBLEM 38.

*In a given circle to inscribe three equal circles, each of which shall touch the other two and the given circle.*

Divide the circumference of the circle into six equal



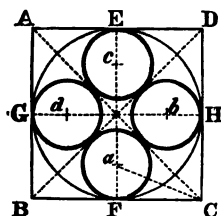
lines already drawn in  $b, c$ ;  $a, b, c$  are the centres of the required circles.

## PROBLEM 39.

*In a given circle to inscribe four equal circles, each touching two others and the given circle.*

Circumscribe the circle by the square  $A B C D$ , and draw the diagonals  $A C, B D$ . Draw the diameters  $E F, G H$ , parallel to  $A B, B C$ .

Fig. 38.



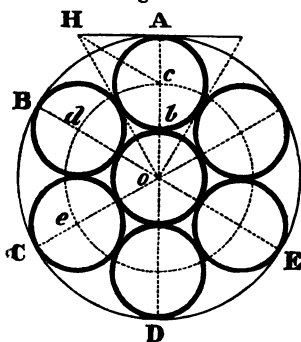
Bisect the angle  $A C B$  by the line  $c a$ , cutting  $E F$  in  $a$ . From the centre of the given circle and from this point to  $a$  as a radius, describe a circle cutting  $E F, G H$  in  $b, c, d$ ; then these points will be the centres and  $a F$  the radius of the required circles.

## PROBLEM 40.

*In a given circle to inscribe seven equal circles touching each other and the given circle.*

Step off the radius of the circle six times in the points

Fig. 39.



$A, B, C$ , etc., and draw  $A D, B E$ , etc. Divide the radius  $o A$  into three parts in the points  $b, c$ . With centre  $o$  and radius  $o c$ , describe a circle, cutting the lines last drawn in  $d, e$ , etc. From these points as centres and with  $A c$  as radius, describe the circles; and from centre  $o$  and same radius draw the central circle.

## PROBLEM 41.

*To inscribe any number of equal circles in a given circle.*

Let the given number of circles be six. Divide the circle (see last Fig.) into six equal parts. Bisect the angle  $\angle o \text{ } \alpha \text{ } \beta$  by  $o \text{ } \eta$ . Draw  $\alpha \text{ } \eta$  a tangent to the circle at  $\alpha$  meeting  $o \text{ } \eta$  in  $\eta$ . Bisect the angle  $\angle \alpha \text{ } \eta \text{ } o$  by  $\eta \text{ } c$ . With centre  $o$  and radius  $o \text{ } c$  describe a circle. With radius  $\alpha \text{ } c$  describe the circles as in the last Problem.

## PROBLEM 42.

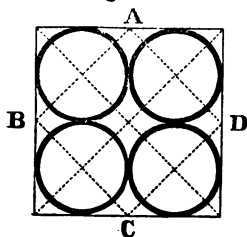
*In a given square to inscribe four equal circles, each touching two others and one side of the square.*

See Fig. 38, Appendix.

## PROBLEM 43.

*In a given square to inscribe four equal circles touching each other, and each of them touching two of the sides of the square.*

Fig. 40.

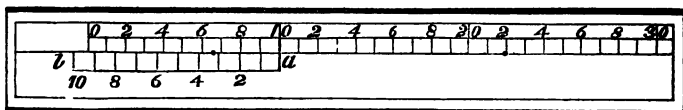


Bisect the sides of the given square in A, B, C, D. Join these points. Draw diagonals of given square. Their points of intersection with the sides of the inscribed square will give the centres of the required circles.

## PROBLEM 44.

*To construct a vernier scale.*

Fig. 41.



Take a line any convenient length, and set off upon it a number of inches. Divide each inch into 10 equal parts and number them 2, 4, 6, 8, 10. At the primary divisions write 0, 10, 20, 30; etc. Take 11 sub-divisions on the natural scale, and set them off from *a* to *b*, and divide the space *ab* into 10 equal parts, number them from right to left 2, 4, 6, etc. Now, the sub-divisions on the vernier scale will exceed those on the natural scale by  $\frac{1}{10}$  of a sub-division or  $\frac{1}{100}$  of a primary division. Taking each sub-division on the natural scale as 10 units, then from *a* to the division 4 will be 44 units,  $= 40 + 4$ , that is, 40 units  $+ \frac{4}{10}$  of 10 units.

The distance between the dots represents 153 or 15·3 or 1·53, according as the primary divisions represent hundreds, tens, or units. Let it be required to take off 195. Place one leg of the compasses on the division on the vernier drawn through 5, and the other leg on the division 4 on the right of the division marked 20. From *a* to the division 5 on the vernier equals  $5 \times 11 = 55$ : and from *a* to the division 4 (on the right of 20) represents 140, which added to 55 gives 195.

SOLUTIONS.

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In the solution of Geometrical problems two methods are employed, *Synthesis* and *Analysis*, an exposition of which we shall now give. Synthesis consists in the building up, step by step, of the different parts of a proposition until we arrive at the desired result. Analysis enables us to deduce from the result the principles or facts on which the result depends.

In illustration of these two methods, let us, for a short time, consider Fig. 36, Prac. Geom. Now, in order to place the circle between the lines  $AB$ ,  $AC$ , the student is told to do certain things, which, when accomplished, give the desired result. This is *synthesis*. Let us now suppose the thing done, our business being to discover *how* it has been effected. The student's manner of going about this will be regulated by the nature and extent of his geometrical knowledge.

However, we will imagine he reasons about the problem thus:—There will be two points, one in  $AB$  and the other in  $AC$ , which will be the same distance from the centre of the circle, and this distance will be the radius of the circle; hence, if we erect a perpendicular to  $AB$ , namely,  $de$ , making it equal to the radius, and from its extremity  $e$ , draw a line parallel to  $AB$ , the centre of the required circle must be somewhere in this line. It must also be in a line bisecting the given angle; it must therefore be where these two lines meet.

Again, if we draw a perpendicular to  $AC$ , making it

equal to the radius of the circle, and draw through its extremity a line parallel to  $A c$ , the centre must be somewhere in this line. But it is also in  $e f$ ; it must therefore be where these lines meet. This is *analysis*—one method building up, the other pulling in pieces.

We would strongly impress upon the student the importance of this process of analysis. As a means of intellectual-geometrical development, it will highly repay him for the labour expended upon it. To aid him in the accomplishment of this end, we would recommend to his consideration the following practical suggestions:—

1st. “As in every other study, endeavour to ascertain what it is you have to do; examine into the nature and meaning of the Proposition, and form a clear, well-defined idea of the quantities concerned in the investigation.

2nd. Construct every figure with exactness, that the eye may aid the judgment.

3rd. It will often be necessary to join certain points so as to form equal triangles, isosceles or equilateral triangles, and other right-lined figures.

4th. Often, also, to lengthen lines, to draw perpendiculars from certain points in a line, or to let fall perpendiculars from certain points on straight lines.

5th. Often, straight lines or angles must be bisected, one angle joined to another, so as to get the sum of two angles; or a line drawn within an angle so as to get the difference of two angles.

6th. Also, it will often be necessary to draw lines parallel to certain lines through remarkable points, which may either be given or required.

7th. If, after trial, he finds he cannot reach the required point, and take the citadel by the path he has sketched out,

he must commence the attack anew by following a different road, and by adopting a different system of tactics."

*Obs.* In the following problems, which we hope to make sufficiently intelligible without the aid of figures, the student must make himself familiar with the symbols which here follow.

## SYMBOLICAL NOTATION.

$\therefore$	because	$\triangle$	triangle
$\therefore$	therefore	$\perp$	perpendicular to
$=$	equal or equals	$\square$	parallelogram
$>$	greater than	$\square$	rectangle
$<$	less than	$\odot$	circle
$\parallel$	parallel	$\odot c$	circumference
$\angle s^*$	angle		

## PROBLEM 45.

*To construct a square having given the diagonal.*

Let  $AC$ , Fig. 32, Appendix, be the given line. Bisect  $AC$  in  $o$ , and draw  $DB \perp$  to it. Make  $oD, oB = oC$ . Join  $AB, BC$ , etc.; and  $ABCD$  is the square required.

## PROBLEM 46.

*The centre of a circle being given, find two opposite points in the circumference by means of a pair of compasses only.*

Given the  $\odot$  with its centre  $A$ . With centre  $B$ , any point in the  $\odot$ , and radius  $AB$ , describe  $\odot$ , cutting the former in  $CD$ . With centre  $C$ , and same radius, describe  $\odot$ , cutting the first in  $E$ . Join  $ED$ ;  $ED$  is the diameter of the first  $\odot$ ;  $\therefore D$  is opposite to  $E$ .

\* When  $s$  is added to a sign it denotes the plural.

## PROBLEM 47.

*From a given isosceles triangle  $ABC$ , cut off a trapezium, which shall have the same base as the triangle, and its remaining three sides equal to each other.*

Bisect  $\angle B, C$  at the base by lines meeting the sides in  $D, E$ . Join  $DE$ .

*Proof.*  $\angle AED = \angle SEBD + \angle EDB$  (Euc. I. 32); and the  $\angle EBD = \frac{1}{2} \angle ABC = \frac{1}{2} \angle AED$ .  $\therefore \angle AED = \angle ABC$  (Euc. I. 29),  $\therefore \angle EDB = \frac{1}{2} \angle AED = \angle EBD$ ; and  $ED = EB = DC$ .

## PROBLEM 48.

*Take a triangle  $ABC$ , and draw a line  $DE$  parallel to the base  $BC$ , so that  $DE$  is equal to the sum of  $BD$  and  $CE$ .*

Bisect the  $\angle$ s at  $B$  and  $C$  by lines meeting in  $F$ . Through  $F$ , draw  $DFE \parallel BC$ .

*Proof.* Then  $\angle DFB = \angle FBC$  (Euc. I. 29)  $= \angle DBF$ ;  $\therefore DF = DB$ . Similarly  $EF = EC$ ;  $\therefore DE = BD + EC$ .

## PROBLEM 49.

*Draw  $DE$  as in last Problem, so that  $DE$  is equal to the difference of  $BD$  and  $CE$ .*

Produce the base  $BC$  to  $F$ . Bisect the  $\angle B$  and exterior  $\angle ACF$  by lines meeting in  $G$ . Through  $G$  draw a line  $\parallel$  to  $BC$ , meeting  $AB$  in  $D$ , and  $AC$  in  $E$ .

*Proof.* The  $\angle DGB = \angle GBC = \angle DBG$ ;  $\therefore DB = DG$ . Similarly  $EG = EC$  (Euc. I. 29); and  $\therefore DE = BD - CE$ .

## PROBLEM 50.

*Find a point in the diagonal produced of a square, from which if a line be drawn parallel to any side of the square, and meeting another side produced, it will form, with the produced diagonal and produced side, a triangle equal to the given square.*

Let  $ABCD$  be the given square. Draw the diagonal  $AC$  and produce it. Produce  $AB$  to  $E$ , and make  $AE = AC$ . From  $E$  draw  $EF \parallel BC$ , meeting  $AC$  produced in  $F$ ;  $F$  is the point sought.

*Proof.* Draw  $EG \perp$  to  $AF$ , meeting it in  $G$ . Then  $\triangle AEF = 2 \triangle AEG = 2 \triangle ABC = ABCD$ .

## PROBLEM 51.

*To trisect a right angle.*

Let  $ABC$  be the right angle. Upon  $BC$  describe an equilateral triangle  $DBC$ . Bisect the angle  $DBC$  by  $BE$ .

*Proof.* Then  $\angle DBC = \frac{1}{2}$  of a right angle ( $= 60^\circ$ );  $\therefore DBA = \frac{1}{2}$  right angle  $= DBE = EBC$ ; and  $\therefore \angle ABC$  is trisected.

## PROBLEM 52.

*Given the perpendicular from the vertex on the base, 1.7 inch, and difference between each side and the adjacent segment on the base, .7 inch and 1.3 inch respectively, to construct the triangle.*

Take any line and any point  $A$  in it. Upon opposite sides of  $A$  make  $AB, AC$ , .7 inch and 1.3 inch respectively. From  $A$  draw  $AD \perp BC$  and  $= 1.7$  inch. Make the  $\angle s CDE, BDF = \angle s DCE, DBF$ , viz.,  $CDE$  to  $DCE$ , and  $BDF$  to  $DBF$ .  $\therefore DE = EC$ ,  $\therefore DE - EA = CA$ ; so also  $DF - FA = BA$ ; and  $DEF$  is the triangle required.

## PROBLEM 53.

*Upon a base 2.4 inches long, construct a triangle  $\triangle ABC$ , the angles at the base being  $40^\circ$  and  $30^\circ$ . Place a line in the triangle parallel to the base and 1 inch long.*

For the first part of the problem, see Fig. 37, Prac. Geom.

Next make  $ba$ , Fig. 27, Appendix, 1 inch. From  $a$  draw  $ab \parallel AB$  and meeting  $ac$  in  $b$ . From  $b$  draw  $bc \parallel$  to  $BC$ ;  $abc$  is the line sought.

*Proof.*  $ba$   $abc$  is a  $\square$ ;  $\therefore bc = ba$ .

## PROBLEM 54.

*Through a given point out-side two parallel lines, draw a line so that the part of it intercepted between the given lines may be 1.5 inch long, the two parallel lines being .5 inch apart.*

Let  $A$  be the given point. In one of the lines take any point  $B$ ; and with  $B$  as a centre and radius 1.5 inch, describe  $\odot$ , cutting the other line in  $C, D$ . Join  $BC, BD$ , and from  $A$  draw lines  $\parallel$  to  $BC, BD$ .

*Proof.* The parts intercepted by either of the lines last drawn, will form with  $BC, BD$ ,  $\square$ s, the opposite sides of which are equal.

## PROBLEM 55.

*Show that in any right-angled triangle, the middle point of the hypotenuse is equally distant from the three angles.*

Let  $\triangle ABC$  be  $\triangle$ ,  $BC$  being the base. Bisect  $AC$  in  $D$ , and from  $D$  draw  $DE, DF \perp AB, BC$ ; join  $BD$ .

*Proof.* In  $\triangle SDAE, DFC$ ,  $AE = DF = EB$ ; and in  $\triangle SDEA, DEB, DB = DA = DC$ .

## PROBLEM 56.

*Find a point in the base  $BC$  of a triangle  $ABC$ , from which lines drawn parallel to the sides to meet them are equal.*

Bisect the  $\angle BAC$  by  $AD$ , meeting the base in  $D$ . From  $D$ , draw  $DE \parallel AC$  and  $DF \parallel AB$ ;  $DE = DF$ .

*Proof.* See Euc. I. 29 and 6.

## PROBLEM 57.

*Show that the perimeter of an isosceles triangle is less than that of any other equal triangle upon the same base.*

(See Fig. 46, Prac. Geom.)

From  $C$  draw  $CF \perp AE$  produced. Produce  $CF$  to  $G$ , making  $FG = FC$ . Join  $GE, GA$ ;  $BEG$  is a straight line.

*Proof.*  $GA + AB$  are greater than  $BG$  (Euc. I. 20).  $CA$  (which is equal  $AG$ )  $+ AB$  are greater than  $BG$ ; and  $\therefore$  greater than  $BE + EC$ ;  $\therefore BE + EC = BG$ .

## PROBLEM 58.

*Given a triangle  $ABC$  and a point  $D$  in  $AB$ , the base, to construct another triangle  $ADE$  equal to the former and having the common angle  $A$ .*

Take  $D$  the given point anywhere in  $AB$ . Join  $DC$ , and from  $B$  draw  $BE \parallel DC$ , meeting  $AC$  produced in  $E$ . Join  $ED$ .

*Proof.*  $\triangle CBD = \triangle CED$  (Euc. I. 38). Add  $\triangle CAD$  to each; and  $\triangle ABC = \triangle ADE$ .

## PROBLEM 59.

*Construct a triangle whose perimeter is 7 inches and whose sides are as  $2:4:5$ .*

Make  $AB$ , Fig. 7, Appendix, 7 inches. Make  $Aa, ab$ ,

$b$   $B$  in the ratio of 2, 4, 5 (Fig. 20, Appendix). With centres  $a$ ,  $b$ , and radii  $a$   $A$ ,  $b$   $B$ , describe arcs intersecting in  $c$ . Join  $c$   $a$ ,  $c$   $b$ ; and  $a$   $b$   $c$  is the triangle required.

#### PROBLEM 60.

*To make a triangle of a given altitude equal in area to a given triangle.*

Let  $ABC$  be the given  $\triangle$ ,  $BC$  being its base. At  $B$  draw  $BD \perp BC$  and = given altitude. Draw  $DE \parallel$  to  $BC$ , to meet  $AB$  produced in  $E$ . Make (Prob. 58, Appendix)  $\triangle BEF = \triangle ABC$ . That is to say, join  $EC$ , and from  $A$  draw  $AF \parallel EC$ . Any  $\triangle$  on  $BF$  as a base, and with its vertex in  $DE$  will be equal to the triangle  $ABC$ .

#### PROBLEM 61.

*Divide a given straight line into two parts, so that the sum of their squares may be the least possible.*

Let  $AB$  be the given line. Bisect  $AB$  in  $C$ . The sum of the squares  $AC$ ,  $CB$  is the least possible.

*Proof.* Divide  $AC$ ,  $CB$  each into 4 equal parts. Then  $AC^2 + CB^2 = 4^2 + 4^2 = 32$ . The squares of any other two divisions, the sum of which =  $AB$ , will be more than 32.

#### PROBLEM 62.

*If the sides of a triangle be as 2, 4, 5, shew whether it will be acute- or obtuse-angled.*

*Proof.* Since  $5^2$  (25) is greater than  $2^2 + 4^2$  (20), the angle opposite the side whose length is 5 will be obtuse.

## PROBLEM 63.

*Divide a given line into two parts so that their rectangle may be equal to a given square.*

Let  $AE$ , Fig. 49, be the given line. At  $E$  draw  $EM \perp AE$ , and  $=$  side of given square. Through  $M$  draw  $MN \parallel AE$ , meeting  $AN$  drawn from  $A \perp AE$  in  $N$ . Upon  $AE$  describe a semi-circle, cutting  $MN$  in  $F$ . Draw  $FB \parallel ME$ . The rectangle contained by  $AB, BE =$  the square of  $FB$ .

## PROBLEM 64.

*Describe three circles 1 inch radius touching one another, and a fourth which shall touch all three.*

Upon a line 2 inches long, describe an equilateral  $\triangle ABC$ . Bisect the sides  $AB, BC, AC$  in  $E, D, F$ . With  $A, B, C$  as centres, and with radius  $AE$  describe three  $\odot$ s. Next bisect  $\angle A, B, C$  by lines meeting in  $O$ , cutting the  $\odot$ s in  $L, M, N$ . With centre  $O$  and radius  $OL$  describe  $\odot$ .

## PROBLEM 65.

*Through a given point  $A$ , within, and .5 inches from the circumference of a circle of 2 inches radius, draw the least possible chord.*

Join  $A$  to  $B$ , the centre of the  $\odot$ . Through  $A \perp AB$ , draw  $CD$ ;  $CD$  is the least possible chord.

*Proof.* If another chord  $EAF$  be drawn, and  $BG$  be drawn  $\perp$  it; then  $BG$  is less than  $BA$ ;  $\therefore CD$  is less than  $EF$ .

## PROBLEM 66.

*Given  $AB$  the base, the vertical angle  $ACD$ , and  $AD$  the perpendicular from the extremity  $A$  of the base  $AB$  on the opposite side  $BC$ , to construct the triangle.*

On  $AB$  describe a segment to contain the given angle (Fig. 29, Prac. Geom.). Next on  $AB$  describe a semi-circle. From  $A$  with the given perpendicular, cut the semi-circle in  $D$ . Join  $BD$  and produce it to meet the  $\odot$ ce of the segment in  $C$ ; join  $AC$ ; and  $\triangle ABC$  is the  $\triangle$  required.

## PROBLEM 67.

*Given a line and a point 4 inches above it, to describe a circle which shall pass through the given point and touch the line.*

Let  $AB$  be the given line and  $c$  the given point. From  $A$  draw  $AD \perp AB$  and  $= 1$  inch. Through  $D$  draw  $DE \parallel AB$ . From centre  $c$  and with radius 1 inch cut  $DE$  in  $F$ ;  $F$  is the centre of the required  $\odot$ .

## PROBLEM 68.

*The hypotenuse of a triangle is 4 inches and the area 2 inches; construct it.*

Make  $AB$  4 inches, and bisect it in  $c$ . From  $c$  draw  $CD \perp AB$  and  $= 1$  inch, and complete the rectangle  $CDEB$ . Upon  $AB$  describe a semi-circle cutting  $DE$  in  $F$ . Join  $FA$ ,  $FC$ ,  $FB$ ;  $\triangle AFB$  is the  $\triangle$  required.

*Proof.*  $\triangle AFB = 2 \triangle CFB = \square CDEB = 2$  inches area. Euc. I. 41.

## PROBLEM 69.

*Given the base, altitude, and vertical angle; to construct the triangle.*

The  $\triangle AEB$ , Fig. 3, Appendix, fulfils the required conditions. Having described upon  $AB$  a segment of a circle to contain  $\angle =$  given  $\angle$ , make  $AC \perp AB =$  altitude, and proceed as shown in the figure.

## PROBLEM 70.

*To make a square which shall be half a given square.*

Upon  $AB$ , Fig. 32, Appendix, construct an isosceles  $\triangle ABE = AOB$ ; then  $AOBE$  is the square required.

## EXAMPLES FOR PRACTICE.

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**NOTE.**—The following questions have been proposed at the different  
Science Examinations.

1. Draw a triangle of 2·75, 3, 3·25 inches sides, and the inscribed and circumscribing circle.

*Solution.* Make  $AB$  (Fig. 37, Prac. Geom.) 3·25 inches. Then with  $A, B$  as centres, and with radii equal to 3 and 2·75 inches, describe arcs cutting in  $C$ . Join  $CA, CB$ . Inscribe the circle by Prob. 22; and circumscribe it by Prob. 7, Prac. Geom.

2. Draw a triangle similar to the foregoing, but of twice its area.

*Solution.* Make  $AB, AC$  each equal to 3·25 inches, and the angle  $BAC$  a right angle. Join  $BC$ ; the triangle described upon  $BC$  shall be equal in area to the two triangles or to twice the area of one of them.

3. Draw a circle of 1 inch radius to touch two lines which meet at an angle of  $30^\circ$ . (Prob. 37, Prac. Geom.)

4. Find a mean proportional to two lines, 3 inches and 1·3 inches long. (Prob. 12, Prac. Geom.)

5. Find a third proportional to two lines 3 and 2 inches long. (Prob. 10, Prac. Geom.)

6. Construct a scale of  $\frac{1}{2}$  to show feet and inches by diagonal division. (See Chapter on Scales.)

7. Construct a triangle whose perimeter is 8 inches and its angles as  $2 : 3 : 4$ . (Fig. 7, Appendix.)

8. In a square 2·5 inches side inscribe one of  $\frac{2}{3}$  the area. (Fig. 49, Appendix.)

9. Draw a rectangle of 3 and 1·3 inches, and construct a square equal to it in area. (Fig. 49, Prac. Geom.)

10. Construct an irregular seven-sided polygon  $A B C D E F G A$ , from either of the following sets of conditions:—

(a) *Sides.*

$$A B = 1\cdot25 \text{ inches.} \quad A G = 1\cdot6 \text{ inches.}$$

$$B C = 2\cdot2 \quad ,, \quad G F = 1\cdot5 \quad ,,$$

$$C D = 2\cdot2 \quad ,, \quad F E = 1\cdot5 \quad ,,$$

$$\text{Diagonal } A E = 3\cdot7.$$

*Angles.*

$$A B C = 155^\circ \quad B A E = 70^\circ$$

$$B C D = 115^\circ$$

$$B A G = 110^\circ$$

Write down the lengths of  $D E$ ,  $C F$ .

(b) *Sides and Diagonals.*

$$A B = 1\cdot25 \text{ inches.} \quad B P \text{ (in } B E) = 2\cdot3 \text{ inches.}$$

$$A E = 3\cdot72 \quad ,, \quad P D = 2\cdot15 \text{ inches.}$$

$$B E = 3\cdot5 \quad ,, \quad P G = 2 \text{ inches.}$$

$$P F = 1\cdot58 \text{ inches.}$$

*Angles.*

$$\left. \begin{array}{l} B P D = 113^\circ \\ B P G = 67^\circ \end{array} \right\} E P F = 65^\circ$$

Write down the angles  $A B E$ ,  $B A G$ .

(Prob. 15, Appendix.)

11. A line  $A B$  is 3 inches, and  $A C$  is 2 inches. Find a point  $D$  in it, such that, (a)  $A B \times C D = A D \times B C$ , or (b) Find a point  $D$  in  $A B$  produced, such that  $A D, C D = A C, B D$ . (Prob. 17, Appendix.)

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